Math 4547: Real Analysis I

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1 January 6, 2025

Professor Margolis introduced the course and discussed the syllabus.

1.1 What is Analysis?

Analysis is the branch of mathematics that deals with the rigorous study of limits, functions, derivatives, integrals, and infinite series. It provides the foundation for calculus and extends its concepts to more abstract settings.

> **Theorem 1**
> Every convergent sequence is bounded.

1.2 The Real Numbers

1.2.1 What are the real numbers?

- The **natural Numbers** $\mathbb{N} = \{1, 2, 3, ...\}$
- The **integers** $\mathbb{Z} = \{0, 1, -1, 2, -2, \cdots \}$
- The **rational Numbers** $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\}$
- The **real Numbers** $\mathbb{R}$
- The **complex Numbers** $\mathbb{C}: = \{ a + bi \mid a, b \in \mathbb{R}\}$, where $i^2 = -1$

> **Theorem 2**
> There is no rational number x, such that $x^2 = 2$.

Proof. We assume for contradiction that such an x exists. Then $x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ and $q \neq 0$. We can assume that p and q have no common factors. Then, $\frac{p^2}{q^2} = 2$, which implies

$$p^2 = 2q^2$$

Thus, p^2 is even. As the square of an odd number is odd, it follows p must even. Therefore, $p = 2k$ for an integer k. We have $2q^2 = p^2 = (2k)^2 = 4k^2$, and so $q^2 = 2k^2$. Thus, q^2 is even. Since p and q are both even, this contradicts our assumption that p and q have no common factors. Therefore, there is no rational number x such that $x^2 = 2$. □

This theorem implies, if we visualize $\mathbb{Q}$ as points lying on a number line, there is a 'hole' where $\sqrt{2}$ is. (There are many more 'holes' e.g. π, e, $\sqrt{3}$, ...)

The key property that $\mathbb{R}$ possesses, but $\mathbb{Q}$ doesn't is that $\mathbb{R}$ has "no holes" (formally, $\mathbb{R}$ is complete.)

In this class, we will rigorously deduce all properties of $\mathbb{R}$ from the axioms of the real numbers.

The axioms are in three groups.

1. Field Axioms (addition and multiplication)
2. Order axioms (needed to describe properties concerning inequalities)
3. Completeness Axiom

1.3 Addition axioms

1. For every pair $a, b \in \mathbb{R}$, we can associate a real number $a + b$ called their **sum**.
2. For every real number a, there is a real number $-a$ called its **negative** or **additive inverse**.
3. There is a special real number 0 called zero or the additive identity such that for all $a, b, c, x, y, z \cdots$ are real numbers unless otherwise stated:
   (a) $a + b = b + a$
   (b) $a + (b + c) = (a + b) + c$
   (c) $a + 0 = a$
   (d) $a + (-a) = 0$

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In this lecture, we will use the axioms to deduce various properties of the real numbers $\mathbb{R}$. From these axioms, we can derive many more properties of the real numbers.

Proposition 3
If $x + a = x$ for all $a \in \mathbb{R}$, then $a = 0$.

Proof. We know that

$$\begin{aligned} x &= x + 0 \quad \text{(A3)} \\ &= x + a \quad \text{(by assumption on } a\text{)} \end{aligned}$$

By the left cancellation property of addition, it follows that $a = 0$. □

Proposition 4 (Left cancellation of addition)
If $a + x = a + y$, then $x = y$.

Proof. We start with the given equation $a + x = a + y$. By the additive identity property (A3), we have:

$$\begin{aligned}
y &= y + 0 && \text{(A3)} \\
&= y + (a + (-a)) && \text{(A4)} \\
&= (y + a) + (-a) && \text{(A2)} \\
&= (a + y) + (-a) && \text{(A1)} \\
&= (a + x) + (-a) && \text{(given)} \\
&= x + (a + (-a)) && \text{(A1)} \\
&= x + 0 && \text{(A4)} \\
&= x && \text{(A3)}
\end{aligned}$$

Therefore, $x = y$. □

Proposition 5
$-(-a) = a$

Proof. We need to show that $-(-a) = a$. Consider the following:

$$\begin{aligned}
(-a) + (-(-a)) &= 0 \quad \text{(by definition of additive inverse)} \\
(-a) + a &= 0 \quad (\text{since } -(-a) = a) \\
a + (-a) &= 0 \quad \text{(by commutativity of addition)} \\
(-a) + (-(-a)) &= a + (-a) \quad \text{(by substitution)} \\
(-(-a)) &= a \quad \text{(by left cancellation of addition)}
\end{aligned}$$

Therefore, $-(-a) = a$. □

Proposition 6
$-(a + b) = (-a) + (-b)$

Proof. We need to show that the additive inverse of $(a + b)$ is equal to the sum of the additive inverses of a

and b. Consider the following:

$$\begin{aligned}
(a+b)+(-(a+b)) &= 0 \quad \text{(by definition of additive inverse)} \\
(a+b)+((-a)+(-b)) &= a+(b+((-a)+(-b))) \quad \text{(by associativity of addition)} \\
&= a+((b+(-a))+(-b)) \quad \text{(by associativity of addition)} \\
&= a+((-a)+(b+(-b))) \quad \text{(by commutativity of addition)} \\
&= a+((-a)+0) \quad \text{(by definition of additive inverse)} \\
&= a+(-a) \quad \text{(by identity property of addition)} \\
&= 0 \quad \text{(by definition of additive inverse)}
\end{aligned}$$

Therefore, $-(a+b)=(-a)+(-b)$. □

Proposition 7

$-0=0$

Proof. We need to show that the additive inverse of 0 is 0. Consider the following:

$$\begin{aligned}
0+0 &= 0 \quad \text{(by the identity property of addition, A3)} \\
0+(-0) &= 0 \quad \text{(by the definition of additive inverse, A4)}
\end{aligned}$$

Therefore, we have:

$$0+0=0+(-0)$$

By the left cancellation property of addition, it follows that:

$$0=-0$$

Therefore, $-0=0$. □

2.1 Multiplication Axioms

Definition 8

For all $a,b\in\mathbb{R}$, we can associate a real number $a\times b$ called their **product**.

Definition 9

For every $a\in\mathbb{R}$, there is some $a^{-1}\in\mathbb{R}$ called its **multiplicative inverse** or **reciprocal** such that for all $a\neq 0$, $a\times a^{-1}=1$.

Definition 10

There is a number 1 called **one** or the **multiplicative identity** such that for all $a \in \mathbb{R}$, $a \times 1 = a$.

Definition 11

For all $a, b, c \in \mathbb{R}$, we have the following properties of multiplication:

- For all $a, b \in \mathbb{R}$, $a \times b = b \times a$.
- For all $a, b, c \in \mathbb{R}$, $a \times (b \times c) = (a \times b) \times c$.
- For all $a \in \mathbb{R}$, $a \times 0 = 0$.
- For all $a, b \in \mathbb{R}$, $a \times (b + c) = a \times b + a \times c$.

Proposition 12

If $a \times b = a$, and $a \in \mathbb{R}$ then $b = 1$.

Proof. We start with the given equation $a \times b = a$. By the multiplicative identity property, we have:

$$\begin{aligned} a \times b &= a \times 1 \\ b &= 1 \quad \text{(by left cancellation of multiplication)} \end{aligned}$$

Therefore, $b = 1$. □

Proposition 13

If $a \neq 0$ and $a \times b = a \times c$, then $b = c$.

Proof. We start with the given equation $a \times b = a \times c$. By the multiplicative inverse property, we have:

$$\begin{aligned} a^{-1} \times (a \times b) &= a^{-1} \times (a \times c) \\ (a^{-1} \times a) \times b &= (a^{-1} \times a) \times c \\ 1 \times b &= 1 \times c \\ b &= c \end{aligned}$$

Therefore, $b = c$. □

Proposition 14

If $a \neq 0$ and $a^{-1} \neq 0$, then $(a^{-1})^{-1} = a$.

Proof. We need to show that the multiplicative inverse of a^{-1} is a. Consider the following:

$$\begin{aligned} a^{-1} \times a &= 1 \quad \text{(by definition of multiplicative inverse)} \\ (a^{-1})^{-1} \times a^{-1} &= 1 \quad \text{(by definition of multiplicative inverse)} \\ (a^{-1})^{-1} &= a \end{aligned}$$

Therefore, $(a^{-1})^{-1} = a$. □

Proposition 15

If $a \neq 0$, $b \neq 0$, and $a \times b \neq 0$, then $(a \times b)^{-1} = a^{-1} \times b^{-1}$.

Proof. We need to show that the multiplicative inverse of $a \times b$ is $a^{-1} \times b^{-1}$. Consider the following:

$$\begin{aligned} (a \times b) \times (a^{-1} \times b^{-1}) &= a \times (b \times (a^{-1} \times b^{-1})) \\ &= a \times ((b \times a^{-1}) \times b^{-1}) \\ &= a \times (1 \times b^{-1}) \\ &= a \times b^{-1} \\ &= 1 \end{aligned}$$

Therefore, $(a \times b)^{-1} = a^{-1} \times b^{-1}$. □

Proposition 16

If $a, b, c \in \mathbb{R}$, then $(a + b) \times c = (a \times c) + (b \times c)$.

Proof. We need to show that the product of $(a + b)$ and c is equal to the sum of the products of a and c, and b and c. Consider the following:

$$\begin{aligned} (a + b) \times c &= c \times (a + b) \\ &= c \times a + c \times b \\ &= a \times c + b \times c \end{aligned}$$

Therefore, $(a + b) \times c = (a \times c) + (b \times c)$. □

Proposition 17

For all $a \in \mathbb{R}$, $a \times 0 = 0$.

Proof. We need to show that the product of any real number a and 0 is 0. Consider the following:

$$\begin{aligned} a \times 0 &= a \times (0 + 0) \\ &= a \times 0 + a \times 0 \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore, $a \times 0 = 0$. □

Proposition 18

If $a \times b = 0$, then either $a = 0$ or $b = 0$ or both.

Proof. We need to show that if the product of a and b is 0, then either a or b or both must be 0. Consider the following:

$$a \times b = 0$$

If $a \neq 0$, then $b = 0$ by the multiplicative inverse property. If $b \neq 0$, then $a = 0$ by the multiplicative inverse property. Therefore, if $a \times b = 0$, then either $a = 0$ or $b = 0$ or both. □

Proposition 19

$a \times (-b) = (-a) \times b$. In particular, $a \times (-1) = -a$.

Proof. We need to show that the product of a and $-b$ is equal to the product of $-a$ and b. Consider the following:

$$\begin{aligned} a \times (-b) + a \times b &= a \times (b + (-b)) \\ &= a \times 0 \\ &= 0 \\ &= a \times b + (-(a \times b)) \end{aligned}$$

Hence, the additive inverse of $a \times b$ is $-(a \times b)$. Therefore, $a \times (-b) = (-a) \times b$. □

Proposition 20

$(-1) \times (-1) = 1$

Proof. We need to show that the product of -1 and -1 is 1. Consider the following:

$$\begin{aligned}(-1)\times(-1) &= -(-1)\times 1 \\ &= -(-1)\times(1+0) \\ &= -(-1)\times(1+(-1)) \\ &= -(-1)\times 0 \\ &= 0 \\ &= (-1)+(-1)\end{aligned}$$

Therefore, $(-1)\times(-1) = 1$. □

3 January 10, 2025

For all $a, b \in \mathbb{R}$, we write:

- ab or $a \cdot b$ for $a \times b$.
- $a - b$ for $a + (-b)$.
- $\frac{1}{a}$ for a^{-1} if $a \neq 0$.
- $\frac{a}{b}$ for ab^{-1} if $b \neq 0$.

For $a \neq 0$, we write:

- a^0 for 1.
- a^{k+1} for $a^k \cdot a$ for $k = 0, 1, 2, \ldots$
- $a^{-1} for (a^l)^{-1}$ for $l = 1, 2, 3$

Definition 21

Any set equipped with operations $+$ and $\times$ staisfying A1 - A4, M1 - M4, Z, D is a **field**.

Fact 22

Some facts about the fields:

- $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ are all fields.
- $\mathbb{Z}$ is not a field (M4 isn't satisified).
- $\mathbb{N}$ is not a field $(A4, M4)$ are not satsified.
- $\frac{\mathbb{Z}}{p\mathbb{Z}}$ (integers mod p for prime p) is a field.

3.1 The order axioms

The order axioms are: There is as subset of $P \subset \mathbb{R}$ called the set of **positive numbers**.

- If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$. (P1)
- If $a, b \in \mathbb{P}$, then $a \times b \in \mathbb{P}$. (P2)
- For each $a \in \mathbb{R}$, exactly one of the following is true: $a \in \mathbb{P}$, $a = 0$, or $-a \in \mathbb{P}$. $\leftarrow$ Law of Trichotomy (P3)

P3 is the most powerful aximom about the positive numbers.

Proposition 23

Prove that $1 \in \mathbb{P}$

Proof. According to **P3**, either

- $1 \in \mathbb{P}$
- $1 = 0$
- $-1 \in \mathbb{P}$

We will prove (b) and (c) are false by contradiction and then show that $1 \in \mathbb{P}$. If **(b)** holds, $1 = 0$, which contradicts **Z**. Assume for contradiction **(c)** holds. We know from last lecture that $1 = -(-1)$. Since $-1 \in \mathbb{P}$, by (P2), $(-1) \times (-1) \in \mathbb{P}$. But $(-1) \times (-1) = 1$, so $1 \in \mathbb{P}$. $\therefore$, $1 \in \mathbb{P}$ and $-1 \in \mathbb{P}$ contradicts **P3**. Since, **(b)** and **(c)** cannot hold, therefore, **(a)** must hold. □

Fact 24

For all $a, b \in \mathbb{R}$, we write

- $a < b$ if $b - a \in \mathbb{P}$
- $a > b$ if $a - b \in \mathbb{P}$
- $a \leq b$ if $b - a \in \mathbb{P} \cup \{0\}$
- $a \geq b$ if $a - b \in \mathbb{P} \cup \{0\}$

Proposition 25

$a > b$ if and only if $-a < -b$. In particular, $x > 0 \iff -x < 0$

Proof.

$$\begin{aligned} a > b &\iff a - b \in \mathbb{P} \\ &\iff -(-(a)) - b \in -\mathbb{P} \\ &\iff -b - (-a) \in -\mathbb{P} \\ &\iff -a < -b \end{aligned}$$

$\square$

Proposition 26

For all $x, y, z \in \mathbb{R}$ the following holds:

- $x \leq x$
- If $x \leq y$ and $y \leq z$, then $x \leq z$.
- If $x \leq y$ and $y \leq z$, then $x \leq z$.

Proof. $\square$

Proposition 27

If $x, t, z \in \mathbb{R}$ and $x < y$, then $x + z < y + z$.

Proof. Since $x < y$, we have $x - y \in \mathbb{P}$. By the properties of addition (A1-A4), we know that:

$$(y + z) - (x + z) = y - x$$

Since $y - x \in \mathbb{P}$, it follows that:

$$(y + z) - (x + z) \in \mathbb{P}$$

Hence, $x + z < y + z$. $\square$

Proposition 28

If $x, y, z \in \mathbb{R}$ and $x < y$ and $z > 0$, then $xz < yz$.

Proof. $zy = zx = z(y - x)$. Now, $z \in \mathbb{P}$ and $y - x \in \mathbb{P}$, therefore $zy - zx \in \mathbb{P}$. Therefore, $xz < yz$. $\square$

Corollary 29

If $x, y, z \in \mathbb{R}$ and $x < y$ and $z < 0$, then $xz > yz$.

Proof. $\square$

Corollary 30

For all, $a \in \mathbb{R}$, $a^2 \geq 0$.

Proof. By P3, either $a > 0, a = 0$ or $g < 0$.

- If $a > 0$, then $a^2 = a \times a > 0$. (P2)
- If $a = 0$, then $a^2 = 0 \geq 0$.

- If $a < 0$, then $-a > 0$ and $(-a)^2 = a^2 > 0$.

$\square$

Proposition 31
If $x \in \mathbb{P}$, then $x^{-1} \in \mathbb{P}$.

Proof. Since, $x \in \mathbb{P}, x \neq 0$. Therefore, x^{-1} exists. By P3, $x^{-1} > 0, x^{-1} = 0$, or $x^{-1} < 0$. If $x^{-1} = 0$, then $1 = x \times x^{-1} = x \times 0 = 0$ [Contradiction] Assume $x^{-1} < 0$. Then $-x^{-1} \in \mathbb{P}$ by P3. Then $x \times (-x^{-1}) \in \mathbb{P}$ by P2. But $x \times (-x^{-1}) = -1$, which contradicts P3 since $-1 \notin \mathbb{P}$. Therefore, $x^{-1} \in \mathbb{P}$. $\square$

Corollary 32
If $x, y \in \mathbb{P}$, and $x < y$, then $\frac{1}{y} < \frac{1}{x}$.

4 January 13, 2025

Homework 1 is due on January 21, 2025.

Definition 33
We define $\max\colon \mathbb{R} \times \mathbb{R} \Rightarrow \mathbb{R}$ by

$$\max(a, b) = \begin{cases} a & \text{if } a \geq b \\ b & \text{if } b \geq a \end{cases}$$

Definition 34
We define $\max\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\max(a, b) = \begin{cases} a & \text{if } a \geq b \\ b & \text{if } b > a \end{cases}$$

Definition 35
We define $|x| \ : \mathbb{R} \to \mathbb{R}$ as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Proposition 36
For all $x \in \mathbb{R}$, $|-x| = |x|$.

Proof. By P3, x > 0, x = 0, or x < 0.

- Case 1: If $x > 0$, then $|x| = x$ and $|-x| = -(-x) = x$. Thus, $|x| = |-x|$.
- Case 2: If $x = 0$, then $|x| = 0$ and $|-x| = -0 = 0$. Thus, $|x| = |-x|$.
- Case 3: If $x < 0$, then $|x| = -x$ and $|-x| = -(-x) = x$. Thus, $|x| = |-x|$.

□

Theorem 37 (The Triangle Δ Inequality)
For all $a, b, \in \mathbb{R}$

$$|a+b| = \leq |a| + |b|$$

with equality if and only if either $a \geq 0$ and $b \geq 0$ or $a \leq 0$ and $b \leq 0$.

Proof. By P3, oen of the following 8 Cases must hold:

	a	b	a + b
1	≥ 0	≥ 0	Row 2, Col 4
2	≥ 0	≥ 0	Row 3, Col 4
3	≥ 0	< 0	Row 4, Col 4
4	≥ 0	< 0	Row 5, Col 4
5	< 0	Row 6, Col 3	Row 6, Col 4
6	< 0	Row 7, Col 3	Row 7, Col 4
7	< 0	Row 8, Col 3	Row 8, Col 4
8	< 0	Row 8, Col 3	Row 8, Col 4

Case 2 and 7 is not possible. But we will prove the rest of the cases:

(1) $|a| = a$, $|b| = b$, $|a+b| = a+b$, $\therefore$ $|a+b| = a+b = |a| + b$

(3)

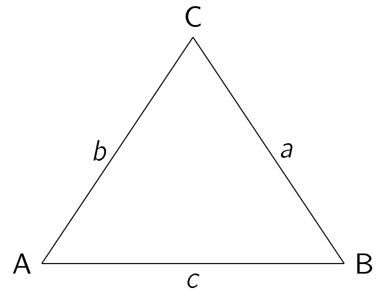
(4) $|a| = a, |b| = -b, |a+b| = a+b$

$$\begin{aligned} |a+b| = -a-b = -0-b = (-a \leq 0) \\ &\leq a + 0 \quad (\text{since } b < 0) \\ &= |a| + |b| \end{aligned}$$

(5) Follows the similarly by symmetry.

Finish this for exercise. □

The picture of the Triangle identity



$$|a| = ||\vec{BC}|| \leq |b| + |c| = ||\vec{AC}|| + ||\vec{AB}||$$

Proposition 38
For all $a, b, \in \mathbb{R}$

$$|ab| = |a| \cdot |b|$$

Proof. If $a = 0$ or $b = 0$, then $|ab| = |0| = 0 = |a| \cdot |b|$ Let's assume that $a \neq 0, b \neq 0$.

- $a > 0, b > 0$ Then P2 implies $ab > 0$ so, $|ab| = ab = |a||b|$
- $a < 0, b > 0$. Then $ab < 0$. Then $|ab| = -ab = (-a)b = |a||b|$
- $a > 0, b < 0$. This follows from case 2 by symmetry.
- $a < 0, b < 0$.Then $ab > 0$. Hence, $|ab| = ab = (-a)(-b) = |a||b|$

□

Theorem 39 (Bernouli's Inequality)
For all $x \in \mathbb{R}$ with $x > -1$ and $n \in \mathbb{N}$, if $n \geq 1$, then

$$(1 + x)^n \geq 1 + nx$$

Proof. We proceed by induction on n.
Base Case: $n = 1 : (1 + x)^1 = 1 + x = 1 + 1 \cdot x$
Inductive Step: Assume

$$(1 + x)^N \geq 1 + Nx$$

We want to show $(1 + x)^{N+1} \geq 1 + (N + 1) \cdot x$. First, since $x > -1$, $x + 1 > 0$.
Multiplying both sides by $x + 1$,

$$\begin{aligned}(1 + x)(1 + x)^N &\geq (1 + Nx)(1 + x) \\ &= 1 + (N + 1)x + Nx^2 \quad \text{(field axioms)} \\ &\geq 1 + (N + 1)x \quad (\text{since } N > 0,\ x^2 > 0)\end{aligned}$$

Hence, $(1 + x)^{N+1} \geq 1 + (N + 1)x$. □

5 January 15, 2025

5.1 The completeness axiom

Let $B \subseteq \mathbb{R}$. We say the following:

- We say $b_1 \subseteq$ is a least element or minimum of B if
  - $b_1 \in B$, and
  - $b_1 \leq b$ for all $b \in B$.

  We write $b_1 = \min B$.
- We say $b_1 \subseteq$ is a least element or minimum of B if
  - $b_1 \in B$, and
  - $b_1 \leq b$ for all $b \in B$.

  We write $b_1 = \min B$.

Example 40

Let $B = \{1, 2, 3\}$. Then $\min B = 1$ and $\max B = 3$.

Proposition 41

Let $B \subseteq R$. The maximum of B (if it exists) is unique. Let Similarly, the minimum of B is unique.

Proof. Suppose $a, b \in \mathbb{R}$ are both maximum of B. Since $a \in B$ and b is a max of B, $a, \leqslant b$. Similarly, $b \leqslant a$.Since $a \leqslant b$ and $b \leqslant a$, $a = b$ □

Definition 42

Let $B \subseteq \mathbb{R}$.

- We say h is a **lower bound** of B if $h \leq b$ for all $b \in B$.
- We say h is an **upper bound** of B if $b \leq h$ for all $b \in B$.

Example 43

Let $B = [1, 2)$. Then 1 is a lower bound of B and 2 is an upper bound of B. Note that 1 is the minimum of B, but 2 is not the maximum of B since $2 \notin B$.

Definition 44

Let $B \in \mathbb{R}$. We say B is

- **bounded above** if there exists an upper bound of B.
- **bounded below** if there exists a lower bound of B.
- **bounded** if there exists an upper bound and a lower bound of B.

Fact 45

Example 46

- $\mathbb{N}$ is bounded below, but not bounded above.
- $(-\infty, 1]$ is bounded above but not bounded below
- $(1, 3)$ is bounded.

5.2 Completeness Axiom

Definition 47

A set $B \subseteq \mathbb{R}$ is said to be **bounded above** if there exists a real number M such that $b \leq M$ for all $b \in B$. The number M is called an **upper bound** of B.

Definition 48

A real number s is called the **supremum** or **least upper bound** of a set $B \subseteq \mathbb{R}$ if:

1. s is an upper bound of B.
2. If u is any upper bound of B, then $s \leq u$.

We denote the supremum of B by $\sup B$.

Theorem 49 (Completeness Axiom)

Every non-empty set $B \subseteq \mathbb{R}$ that is bounded above has a supremum.

Example 50

$2 = \sup([1, 2])$

Proof. Suppose, t is an upper bound of B. Suppose towards a contradiction that $t < 2$. Then

$$1 \leqslant t < \frac{t+2}{2} < \frac{2+2}{2} = 2$$

So, $\frac{t+2}{2}$ is an element of the set B that is strictly bigger than t but we arrived to a contradictiopn. Therefore, $2 \leqslant t$. Since 2 is an upper bound of $B = \sup(B)$. □

Example 51

The empty set $\phi \in \mathbb{R}$ has no supremeum since $\mathbb{R}$ is the set of upper bounds of ϕ and $\mathbb{R}$ has no leas element.

Proposition 52

If $B \in \mathbb{R}$ and $\max(B)$ exists then $\max(B) = \sup(B)$.

Proof. Let $A = \max(B)$. Then B is non-empty and bounded is above by a. If $C = \sup(B)$, then $a \leq c$. Since c is an upper bound $a \in B$. Also $c \leqslant a$. Since, c is the least upper bound. $\therefore a = c$ □

Proposition 53 (The approximation property of suprema)

Let $B \subseteq \mathbb{R}$ be non-empty and bounded above. For any $\epsilon > 0$, there exists an element $b \in B$ such that $\sup B - \epsilon < b \leq \sup B$.

Proof. Suppose for contradicton there was some $\epsilon > 0$ such that no b as above exists. Then for all $c \in B, c \leqslant \sup(B) - \epsilon$ i.e. $\sup(B) - \epsilon$ is an upper bound of B. But $\sup(B)$ is the least upper bound of B, so $\sup(B) - \epsilon < \sup(B)$. This is a contradiction. Therefore, there must exist some $b \in B$ such that $\sup(B) - \epsilon < b \leq \sup(B)$. □

Remark 54. *To prove a is a supremum of B, we need show the following:*

- *a is an upper bound of B.*
- *If c is an upper bound of B, then $a \leqslant c$.*

Theorem 55

Suppose, $F \subseteq \mathbb{R}$ is non-empty and bounded below. Then, there exist a **greatest lower bound** of $\mathbb{F}$, called **infimum** of F, denoted $\inf(F)$.

Proof. Let $B = \{x \in \mathbb{R} | -x| \in F\}$. We will show:

- B is bounded above and non-empty, hence $a = \sup(B)$ exists.
- $-a$ is a lower bound of F.
- If c is a lower bound of F, then $c \leqslant -a$.

- Since F is non-empty, B is non-empty. Suppose c is a lower bound of F. Let $x \in B$. Then $-x \in F$, so $-x \geqslant c \to x \leqslant -c$. Hence, c is an upper bound of B. Therefore, B is bounded above.
- Let $f \in F$. Then $-f \in B$, so $-f \leq a$ (since $a = \sup(B)$). Therefore, $f \geq -a$. Hence, $-a$ is a lower bound of F.
- Let c be a lower bound of F. Let $b \in B$. Then $-b \in F$, so $-b \geq c \to b \leq -c$. Hence, $-c$ is an upper bound of B. Therefore, $-a \leq -c \to c \leq -a$.

□

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Corollary 56

Let F be a non-empty and bounded below for each $\epsilon > 0$, there exists $f \in F$ such that

$$\inf(F) \leqslant f < \inf(F) + \epsilon$$

Theorem 57

There is a unique positive number α such that $\alpha^2 = 2$.

Proof. Let $E = \{x \in \mathbb{R} \big| x^2 \big| < 2\}$. Since, $x^1 = 1 < 2$, so $1 \in E$, so $E \neq 0$.
Suppose, $x \geqslant 2$. Then $x^2 \geqslant 2^2 = 4 > 2$. Hence, $x \notin E$. Therefore, if $x \in E$, then $x < 2$.
Hence, $E \neq 0$, and bounded above by 2.
Let $\alpha = \sup(E)$.
We know that $1 \leqslant \alpha \leqslant 2$.
If $\alpha^2 \neq 2$, then either $\alpha^2 = 2$, then either $\alpha^2 > 0$ or $\alpha^2 < 2$. We'll show that both these cases lead to contradiction!

Case 1: When $\alpha^2 < 2$ Let $h = \frac{1}{2}\min(\alpha, \frac{2-\alpha^2}{3\alpha})$. Note $h > 0, h < \alpha$ and $h < \frac{2-\alpha^2}{3\alpha}$. Note that $(\alpha + h)^2 = \alpha^2 + 2\alpha h + h^2 < \alpha^2 + 3\alpha h < \alpha^2 + 3\alpha\frac{2-\alpha^2}{3\alpha} = 2$. So $\alpha + h \in E$. Since $\alpha + h > \alpha$, this contradicts $\alpha = \sup(E)$.

Case 2: When $\alpha^2 > 2$ We set $h = \frac{1}{2}\frac{\alpha^2-2}{2\alpha} > 0$. Since $\alpha - h < \alpha$, the approximation property says that there exists $e \in E$ such that

$$\alpha - h < e \leqslant \alpha$$

. Then $(\alpha - h)^2 < e^2$. Then $\alpha^2 - 2\alpha h + h^2 < 2 \Rightarrow \alpha^2 - 2\alpha h < 2$. Therefore, $h > \frac{\alpha^2-2}{2\alpha}$. And we arrived at a contradiction. Since $\alpha^2 < 2$ and $\alpha^2 > 2$ cannot hold, so $\alpha^2 = 2$.

Suppose $\alpha^2 = \beta^2 = 2$ and $\alpha, \beta > 0$. Then $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = 0$. Since $\alpha + \beta > 0, \alpha - \beta = 0$. Hence, $\alpha = \beta$ □

Remark 58. *The same proof shows $\{\alpha \in \mathbb{Q} \mid \alpha^2 = 2\}$ cannot have a supremum in $\mathbb{Q}$, i.e. completeness doesn't hold.*

Remark 59. *We denote α as above using $\sqrt{2}$ by modifying the previous proof.*

Theorem 60

For any positive real number x, there exist unique postitive real number, denoted $\sqrt{2}$, such that $(\sqrt{x})^2 = x$.

Theorem 61 (Archimedean Property for Real Numbers)

Let $x \in \mathbb{R}$. Then there exists a natural number $n \in \mathbb{N}$ such that $n > x$.

Proof. Suppose this theorem is not true for some $x \in \mathbb{R}$. Then x is an upper bound of $\mathbb{N}$. Since $\mathbb{N} \neq 0, \alpha = \sup(\mathbb{N})$ exists. By approximation theorem, there exists some $n \in \mathbb{N}$ such that

$$\alpha - 1 < n \leqslant \alpha$$

. Hence, $\alpha < n + 1$. Since $n + 1$ is a natural number, this contradicts $\alpha = \sup(\mathbb{N})$ □

Corollary 62

Let $\epsilon > 0$. Then there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Proof. Apply the Archimedian property to $\frac{1}{\epsilon}$ □

Definition 63

A **sequence** is a function $a \colon \mathbb{N} \to \mathbb{R}$. We denote the nth term of the sequence as a_n. We usually write $\alpha(n)$ as α_n

We write α as $((\alpha_n)_{n=1})^\infty$ or simply as α_n. Let a_n and b_n be sequencces and $c \in \mathbb{R}$ then we define a_n

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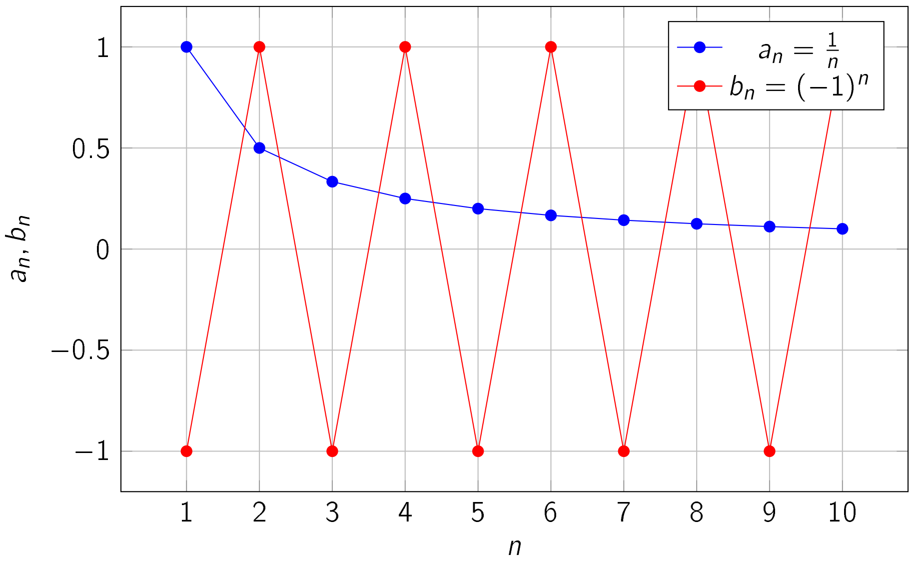
Definition 64

A sequence a_n converges to some $L \in \mathbb{R}$ or (a_n) tends to L, $a_n \to L$, $\lim_{n\to\infty=L}$, $\lim a_n = L$ if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geqslant N |a_n - L| < \epsilon$

Example 65

Let $a_n = \frac{1}{n}$, $\epsilon = \frac{1}{1000}$ For all, $n \geqslant 1000$, $|a_n - 0| = |a_n| = \frac{1}{n} < \frac{1}{1000}$



Definition 66

We say that a sequence (a_n) is convergent if it converges for some $L \in \mathbb{R}$. i.e.

$$\exists L \in \mathbb{R}, \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geqslant N, | \ a_n - L \ | < \epsilon$$

Definition 67

If $a_n \to L$, then L is a limit of a_n.

Definition 68

A sequence (a_n) is divergent if it is not convergent, i.e.

$$\forall L \in \mathbb{R}, \exists \epsilon > 0, \forall N \in \mathbb{N}$$

such that $|a_n - L| \geqslant \epsilon$

Example 69

Let $a_n = \frac{2^n - 1}{2^n}$. Then $a_n \to 1$

Proof. Let $\epsilon > 0$. Then $|a_n - 1| = |\frac{2^n-1}{2^n}| = |\frac{-1}{2^n}| = \frac{1}{2^n}$ We want to prove that $\exists N \in \mathbb{N}$ such that $\forall n \geqslant N, |a_n - 1| < \epsilon$ By Bernouli's Inequality, $2^n = (1+1)^n$. By Archimedean property, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. $\forall n \geqslant N, |a_n - 1| = \frac{1}{2^n} < \frac{1}{n} \leqslant \frac{1}{N} < \epsilon$. Hence, $a_n \to 1$ □

Given a_n and $L \in \mathbb{R}$, players A and B a game.

Example 70

Let

$$a_n = \frac{n^2 + n + 1}{3n^2 + 4}$$

. Then a_n is convergent.

First we need to come up the limit of this sequence.

$$a_n = \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{3 + \frac{4}{n^2}}$$

So $\frac{1}{3}$ seems like a choice of limit L.

Proof. Let $\epsilon > 0$. Then

$$\left| a_n - \frac{1}{3} \right|$$

By the Archimedean property, $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Hence for all, $n \geqslant N$, $\left| a_n - \frac{1}{3} \right| < \frac{1}{n} < \frac{1}{N} < \epsilon$ □

Example 71

Let

$$a_n = \frac{(-1)^n n^2}{n^2 + 1}$$

. Then a_n is divergent.

For large even n,

$$a_n = \frac{n^2}{n^2 + 1} = \frac{1}{1 + \frac{1}{n^2}}$$

For large odd n,

$$a_n = \frac{-n^2}{n^2 + 1} = -\frac{1}{1 + \frac{1}{n^2}}$$

8 January 24, 2025

Definition 72

Let (a_n) be a sequence, and let k be a natural number. Then the kth tail of (a_n) is the sequence $n \mapsto a_{n+k}$, i.e. it equals the sequence $(a_{k+1}, a_{k+2}, a_{k+3}, \ldots)$ which we will also write as $(a_{n+k})_{n=1}^{\infty}$ or $(a_n)_{k+1}^{\infty}$. The tails give a way of focusing on the long-term behavior of a sequence ignoring any short-term behavior at the start of a sequence. Whether a sequence converges purely depends on the long-term behavior of the sequence as we see in the next proposition.

Proposition 73

Let (a_n) be a sequence and let $L \in \mathbb{R}$. Then the following three statements are equivalent.

(i) (a_n) converges (to L);

(ii) some tail of (a_n) converges (to L);

(iii) all tails of (a_n) converge (to L).

Proof. We shall demonstrate the implications as (i) implies (iii), (iii) implies (ii) and (ii) implies (i).

(i) $\implies$ (iii) Suppose that (a_n) converges to L and let $k \in \mathbb{N}$, $\epsilon > 0$. As $(a_n) \to L$ then there exists N such that

$$\forall n \geq N \quad |a_n - L| < \epsilon$$

For such n, we have $n + k \geq n \geq N$ and so

$$\forall n \geq N \quad |a_{n+k} - L| < \epsilon$$

Hence, for any $k \in \mathbb{N}$, the kth tail of (a_n) converges to L.

(iii) $\implies$ (ii) This is obvious.

(ii) $\implies$ (i) Suppose that the kth tail of (a_n) converges to L. Let $\epsilon > 0$. Then there exists N such that

$$\forall n \geq N \quad |a_{n+k} - L| < \epsilon$$

Hence

$$\forall n \geq N + k \quad |a_n - L| < \epsilon$$

and we see that (a_n) converges to L. $\square$

Theorem 74 (Uniqueness of Limits)

Let (a_n) be a real sequence and suppose that $a_n \to L_1$ and $a_n \to L_2$ as $n \to \infty$. Then $L_1 = L_2$.

Proof. Suppose not and set $\epsilon = |L_1 - L_2| > 0$. Then $\epsilon/2 > 0$ and there exists N_1 such that

$$n \geq N_1 \implies |a_n - L_1| < \epsilon/2$$

Likewise there exists N_2 such that

$$n \geq N_2 \implies |a_n - L_2| < \epsilon/2$$

Then for $n \geq \max(N_1, N_2)$ we have

$$\begin{aligned} |L_1 - L_2| &= |(L_1 - a_n) + (a_n - L_2)| \\ &\leq |L_1 - a_n| + |a_n - L_2| \quad \text{by the triangle inequality} \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

which is the required contradiction. $\square$

9 January 27, 2025

Theorem 75 (Limits respect Weak Inequalities)
Let (a_n) and (b_n) be sequences such that $(a_n) \to L$ and $(b_n) \to M$. If $a_n \geq b_n$ for all n, then $L \geq M$.

Proof. Suppose, for a contradiction, that $L < M$. Set $\epsilon = \frac{M-L}{2} > 0$. Since $a_n \to L$, there exists N_1 such that $n \geq N_1 \implies |a_n - L| < \epsilon$. Similarly, since $b_n \to M$, there exists N_2 such that $n \geq N_2 \implies |b_n - M| < \epsilon$. Therefore, for $n \geq \max(N_1, N_2)$, we have:

$$a_n < L + \epsilon = \frac{L+M}{2} \quad \text{and} \quad b_n > M - \epsilon = \frac{L+M}{2}$$

Hence, for $n \geq \max(N_1, N_2)$, we have $a_n < \frac{L+M}{2} < b_n$, which contradicts $a_n \geq b_n$ for all n. ☐

Remark 76. *Clearly,* lim *does not respect strict inequalities: e.g.* $\frac{1}{n} > 0$ *for all* $n \geq 1$ *but* $0 = \lim \frac{1}{n} > \lim 0 = 0$ *is false.*

Theorem 77 (Squeeze Theorem)
Suppose that $x_n \leq a_n \leq y_n$ for all n and that $L = \lim x_n = \lim y_n$. Then $a_n \to L$ as $n \to \infty$.

Proof. Let $\epsilon > 0$. Then there exist N_1 and N_2 such that $x_n - L > -\epsilon$ for all $n \geq N_1$, and $y_n - L < \epsilon$ for all $n \geq N_2$. So for $n \geq \max(N_1, N_2)$, we have:

$$-\epsilon < x_n - L \leq a_n - L \leq y_n - L < \epsilon$$

which shows that $a_n \to L$ also. ☐

9.1 The Algebra of Limits

Most sequences can be built up from simpler ones using addition, multiplication, etc. The algebra of limits tells us how the corresponding limits behave. Throughout the following, (a_n) and (b_n) denote sequences.

Proposition 78 (AOL: Constants)
If $a_n = a$ for all n, then $a_n \to a$.

Proof. For any $\epsilon > 0$, take $N = 1$; $n \geq N \implies |a_n - a| = 0 < \epsilon$. ☐

Proposition 79 (AOL: Sums)
If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.

Proof. Let $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$ and so:

$$\exists N_1 : n \geq N_1 \implies |a_n - a| < \frac{\epsilon}{2}, \quad \exists N_2 : n \geq N_2 \implies |b_n - b| < \frac{\epsilon}{2}$$

Put $N_3 = \max(N_1, N_2)$. Then for all $n \geq N_3$, we have:

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Proposition 80 (AOL: Scalar Products)

If $a_n \to a$ as $n \to \infty$ and $\lambda \in \mathbb{R}$, then $\lambda a_n \to \lambda a$.

Proof. Let $\epsilon > 0$. Then $\frac{\epsilon}{|\lambda|+1} > 0$ and so there exists N such that $|a_n - a| < \frac{\epsilon}{|\lambda|+1}$ for all $n \geq N$. Hence:

$$|\lambda a_n - \lambda a| = |\lambda||a_n - a| \leq \frac{|\lambda|\epsilon}{|\lambda| + 1} < \epsilon$$

for all $n \geq N$. □

Corollary 81 (AOL: Differences)

If $a_n \to a$ and $b_n \to b$, then $a_n - b_n \to a - b$.

Corollary 82 (AOL: Translations)

If $a_n \to a$ and $c \in \mathbb{R}$, then $a_n + c \to a + c$.

Lemma 83

If $x_n \to 0$ and $y_n \to 0$, then $x_n y_n \to 0$.

Proof. Given $\epsilon > 0$, let $\epsilon_1 = \min(1, \epsilon) > 0$. Then:

$$\exists N_1 : n \geq N_1 \implies |x_n| < \epsilon_1, \quad \exists N_2 : n \geq N_2 \implies |y_n| < \epsilon_1$$

So if $n \geq \max(N_1, N_2)$, we have:

$$|x_n y_n| \leq |x_n||y_n| < \epsilon_1^2 \leq \epsilon$$

which completes the proof. □

Proposition 84 (AOL: Products)

If $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.

Proof. Note that:

$$a_n b_n - ab = (a_n - a)(b_n - b) + b(a_n - a) + a(b_n - b)$$

By the previous lemma, $(a_n - a)(b_n - b) \to 0$, and by Proposition 2.2.3, $b(a_n - a) \to 0$ and $a(b_n - b) \to 0$. Hence $a_n b_n \to ab$ by Proposition **??**. □

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Proposition 85 (AOL: Reciprocals)
If $a_n \to a$ and $a_n \neq 0$ for all n and $a \neq 0$, then $\frac{1}{a_n} \to \frac{1}{a}$.

Proof. Let $\epsilon > 0$. As $a \neq 0$ then $\frac{|a|}{2} > 0$. So there exists N_1 such that for $n \geq N_1$ we have $|a_n - a| < \frac{|a|}{2}$. By the Triangle Inequality

$$|a| \leq |a_n| + |a - a_n| = |a_n| + |a_n - a|$$

and so $|a_n| > \frac{|a|}{2}$ and $\left|\frac{1}{a_n}\right| < \frac{2}{|a|}$. Further, as $\frac{|a|\epsilon}{2} > 0$ then there exists N_2 such that for $n \geq N_2$

$$|a_n - a| < \frac{|a|\epsilon}{2}.$$

Set $N_3 = \max(N_1, N_2)$ so that for $n \geq N_3$

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a_n - a|}{|a_n||a|} < \frac{\frac{|a|\epsilon}{2}}{\frac{|a|}{2}|a|} = \epsilon.$$

□

Corollary 86 (AOL: Quotients)
If $a_n \to a$, $b_n \to b$, and $b_n \neq 0$ for all n and $b \neq 0$, then $\frac{a_n}{b_n} \to \frac{a}{b}$.

Proposition 87 (AOL: Modulus)
If $a_n \to a$ then $|a_n| \to |a|$.

Proof. Exercise. □

Example 88
$a_n = \frac{n^2+n+1}{3n^2+4} \to \frac{1}{3}$

Proof. We write

$$\frac{n^2 + n + 1}{3n^2 + 4} = \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{3 + \frac{4}{n^2}} \to \frac{1 + 0 + 0}{3 + 0} = \frac{1}{3}$$

noting

- $\frac{1}{n} \to 0$ by the Archimedean Property;
- $\frac{1}{n^2} \to 0$ by Proposition 2.2.7;
- $1 \to 1$;
- $1 + \frac{1}{n} + \frac{1}{n^2} \to 1$ by Proposition 2.2.2;
- $3 + \frac{4}{n^2} \to 3$ by Proposition 2.2.2;
- $\frac{1}{3+\frac{4}{n^2}} \to \frac{1}{3}$ by Corollary 2.2.9;
- $a_n \to \frac{1}{3}$ by Proposition 2.2.7.

□

Example 89

Suppose $a_1 = 1$, $a_2 = 1$, and we recursively define $a_{n+2} = a_{n+1} + a_n$, for $n \geq 1$.

Proposition 90

$(\frac{a_{n+1}}{a_n})$ is convergent.

Proof. By induction, $a_n \geq 1$ for all n. So for $n \geq 1$

$$\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{a_{n+1}}{a_n}.$$

Write $x_n = \frac{a_{n+1}}{a_n}$ for $n \geq 1$. Note that $x_n > 0$ for all n. Then $x_1 = 1$ and $x_{n+1} = 1 + \frac{1}{x_n}$. Suppose that we did have convergence and that $x_n \to x$. Since x_{n+1} is a tail of x_n, we see that $(x_{n+1}) \to x$ and so $1 + \frac{1}{x_n} \to 1 + \frac{1}{x}$ by AOL.

$$x = 1 + \frac{1}{x}$$

by the Uniqueness of Limits. Hence $x^2 - x - 1 = 0$ giving $x = \frac{1 \pm \sqrt{5}}{2}$. But $x_n \geq 0$ for all n, and so $x \geq 0$ by Theorem 2.1.10 giving

$$x = \frac{1+\sqrt{5}}{2} > 1.$$

At this point we don't yet know the sequence is convergent. We only know that if the sequence converges, then it must converge to the real number $\frac{1+\sqrt{5}}{2}$, which we will denote ϕ. Now we will show that (x_n) does indeed converge to ϕ. ϕ is called the Golden Ratio.

$$x_{n+1} - \phi = 1 + \frac{1}{x_n} - \phi = 1 + \frac{1}{x_n} - 1 - \frac{1}{\phi} = \frac{1}{x_n} - \frac{1}{\phi} = \frac{x_n - \phi}{x_n \phi}$$

as $\phi^2 = \phi + 1$. So

$$\frac{x_{n+1} - \phi}{x_n - \phi} = \frac{1}{|x_n||\phi|} = \frac{1}{\phi x_n} \leq \frac{1}{\phi}$$

as $x_n \geq 1$ for all n. By induction, we get

$$-\frac{1}{\phi^n} \leq x_n - \phi \leq \frac{1}{\phi^n}.$$

and are done by the Squeeze Theorem, since $\phi > 1$ and so $(\pm\frac{1}{\phi^n}) \to 0$. $\square$

11 January 31, 2025

Proposition 91

Let a be a real number with $a > 1$, and $k \in \mathbb{N}$. Then there exists a positive real number c such that $a^n \geq cn^k$ for $n = 1, 2, 3, \ldots$

Proof. Let $a = 1 + b$ so that $b > 0$. Take $n > k$, so that $n \geq k + 1$. We note that $\frac{n-k}{n} = 1 - \frac{k}{n}$ and note that

$$n \geq k + 1 \implies \frac{n-k}{n} \geq \frac{1}{k+1}$$

By the Binomial Theorem,

$$a^n = (1+b)^n = 1 + \binom{n}{1}b + \binom{n}{2}b^2 + \cdots + \binom{n}{k}b^k + \cdots + b^n \geq \binom{n}{k}b^k$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!}b^k > \frac{(n-k)^k}{k!}b^k = \frac{b^k}{k!}\left(\frac{n-k}{n}\right)^k n^k \geq \frac{b^k}{k!(k+1)^k}n^k$$

We have thus found $c_0 > 0$, such that $\frac{a^n}{n^k} \geq c_0$ for $n > k$. If we set

$$c = \min\left(a_1, \frac{a_2}{2^k}, \ldots, \frac{a_k}{k^k}, c_0\right) > 0$$

then, $a^n/n^k \geq c$ holds for $n \geq 1$. $\square$

Corollary 92

Let $a > 1$ and let $k \in \mathbb{N}$. Then $\left(\frac{n^k}{a^n}\right) \to 0$.

Proof. Let $\epsilon > 0$. By Proposition **??**, there exists $c > 0$ such that $a^n \geq cn^{k+1}$ for all $n \in \mathbb{N}$. Hence

$$\left|\frac{n^k}{a^n}\right| \leq \left|\frac{n^k}{cn^{k+1}}\right| = \frac{1}{cn}.$$

By the Archimedean property, there exists some $N \in \mathbb{N}$ such that $N > \frac{1}{c\epsilon}$. Hence for all $n \geq N$, we have

$$\left|\frac{n^k}{a^n}\right| \leq \frac{1}{cn} \leq \frac{1}{cN} < \epsilon.$$

$\square$

12 February 3, 2025

Example 93

Let $a_n = n$ and $b_n = (2n+1)^2$. Then (a_n) and (b_n) are strictly increasing.

Definition 94

We say that a sequence (a_n) is:

1. bounded above if the set $S = \{a_n \mid n \in \mathbb{N}\}$ is bounded above.
2. bounded below if the set $S = \{a_n \mid n \in \mathbb{N}\}$ is bounded below.
3. bounded if the set $S = \{a_n \mid n \in \mathbb{N}\}$ is bounded.

Theorem 95 (Monotone Convergence Theorem)

The power of this theorem is that we can show a sequence converges without first calculating its limit.

1. Let (a_n) be an increasing, bounded above sequence. Then (a_n) converges.
2. Let (a_n) be a decreasing, bounded below sequence. Then (a_n) converges.

Proof. If (a_n) is decreasing and bounded below, then $(-a_n)$ is increasing and bounded above, so it suffices to prove (1). We thus assume (a_n) is increasing and bounded above. Let $L = \sup\{a_n \mid n \in \mathbb{N}\}$; this exists by the completeness axiom as the set is bounded above and non-empty. Let $\epsilon > 0$. By the Approximation Property there exists $N \in \mathbb{N}$ such that

$$L - \epsilon < a_N \leq L.$$

As the sequence is increasing, then for any $n \geq N$

$$L - \epsilon < a_N \leq a_n \leq L,$$

which implies $\forall n \geq N \mid a_n - L \mid < \epsilon$. That is $(a_n) \to L$. □

Corollary 96

Let (a_n) be a bounded monotone sequence. Then (a_n) converges.

Example 97

Let $x > 0$. Set $a_1 = 1$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{x}{a_n}\right)$ for each n. Then (a_n) converges to some positive number a such that $a^2 = x$.

Proof. By induction, it follows that a_n is defined and $a_n > 0$ for all $n \geq 1$. We remark that for all n, we have $2a_na_{n+1} = a_n^2 + x$. Therefore,

$$0 \leq (a_n - a_{n+1})^2 = a_n^2 - 2a_na_{n+1} + a_{n+1}^2 = a_{n+1}^2 - x,$$

and so $a_{n+1}^2 \geq x$ for all $n \geq 1$. We now show that (a_{n+1}) is decreasing. Indeed, for $n \geq 2$,

$$a_n - a_{n+1} = a_n - \frac{1}{2}\left(a_n + \frac{x}{a_n}\right) = \frac{a_n^2 - x}{2a_n} \geq 0.$$

Therefore, $a_n \geq a_{n+1}$ for all $n \geq 2$. Therefore the sequence (a_{n+1}) is decreasing and bounded below by 0, hence converges to some $a \geq 0$. Since $a_{n+1}^2 \geq x$ for all $n \geq 1$, AOL implies $a^2 \geq x > 0$, and so $a > 0$. Moreover, since (a_{n+1}) is a tail of (a_n), it follows that (a_n) also converges to a. Since $a_{n+1} = \frac{1}{2}\left(a + \frac{x}{a}\right)$, $a > 0$ and $a > 0$, AOL implies that

$$a = \frac{1}{2}\left(a + \frac{x}{a}\right).$$

Rearranging gives $x = a^2$, as required. □

12.1 Subsequences

Example 98

Let $a_n = \frac{1}{n^2}$, i.e.

$$(a_n) = \left(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\right).$$

We can get new sequences by looking at:

- everything after the second place $\left(\frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\right)$
- all odd terms $\left(1, \frac{1}{9}, \frac{1}{25}, \ldots\right)$
- all prime terms $\left(\frac{1}{4}, \frac{1}{9}, \frac{1}{25}, \ldots\right)$

Definition 99

Let (a_n) be a sequence. We say that a sequence (b_n) is a subsequence of (a_n) if there is a strictly increasing sequence of natural numbers $(f(n))$ such that $(b_n) = (a_{f(n)})$. (There may be more than one such function f.) Often we write n_r for $f(r)$ and write a subsequence as (a_{n_r}) or $(a_{n_r})_{r=1}^{\infty}$.

Example 100

Let

$$(a_n) = \left(n^2\right) = (1, 4, 9, 16, \ldots)$$

$$(b_n) = (0) = (0, 0, 0, 0, \ldots)$$

$$(f(n)) = (2n) = (2, 4, 6, 8, \ldots)$$

$$(g(n)) = (2n-1) = (1, 3, 5, 7, \ldots)$$

Then

$$(a_{f(n)}) = (a_{2n}) = (4, 16, 36, 64, \ldots)$$

$$(a_{g(n)}) = (a_{2n+1}) = (1, 9, 25, 49, \ldots)$$

$$(b_{f(n)}) = (b_{2n}) = (0, 0, 0, 0, \ldots)$$

$$(b_{g(n)}) = (b_{2n+1}) = (0, 0, 0, 0, \ldots)$$

Proposition 101

Suppose that the sequence (a_n) converges to L. Then every subsequence (a_{n_r}) also converges to L.

Proof. Let $\epsilon > 0$. Then there exists N such that

$$n \geq N \implies |a_n - L| < \epsilon.$$

As $r \mapsto n_r$ is increasing then $n_r \geq r$ for all r and so

$$r \geq N \implies n_r \geq N \implies |a_{n_r} - L| < \epsilon.$$

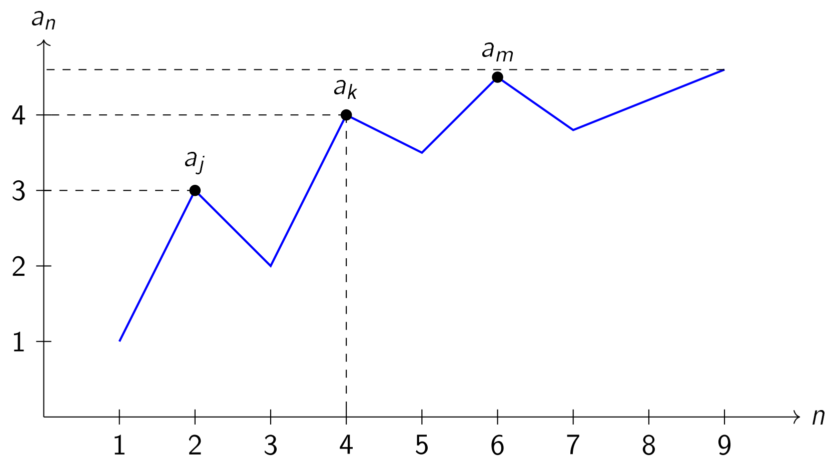
□

13 February 5, 2025

Theorem 102 (Monotone Subsequence Theorem)

Let (a_n) be a sequence. Then (a_n) has a monotone subsequence which means the subsequence is either increasing or decreasing over that bound.

Definition 103 (Vantage Point)



Here, a_k, a_m, and a_j are vantage points.

Proof. We say that $k \in \mathbb{N}$ is a 'vantage point' of (a_n) if $a_m < a_k$ for all $m > k$. Let $V = \{k \in \mathbb{N} \mid k \text{ is a vantage point}\}$.

- **Case 1: V is infinite:**
  Let $V = \{k_1, k_2, \dots\}$ where $k_1 < k_2 < k_3 < \dots$. Each k_i is a vantage point. If $m > n$, then $a_{k_m} < a_{k_n}$. Hence $(a_{k_n})_{n=1}^{\infty}$ is a decreasing subsequence.

- **Case 1: V is finite:**
  There exist k_1 such that $\forall n \geq k_1, a_n$ is not a vantage point, $\exists k_2 > k_1$ such that $k_2 > k_1$) Since, $k_2 > k_1$ is also not a vantage point. hence $\exists k_3 > k_2$ such that $a_{k_3} \geq a_{k_2}$.
  We continue in this way, defining a strictly increasing subsequence.

$$k_1 < k_2 < k_3 < \dots$$

such that

$$a_{k_1} \leq a_{k_2} \leq a_{k_2} \leq a_{k_3} \leq \dots$$

. Hence $(a_{k_n})_{n=1}^{\infty}$ is an increasing subsequence. Therefore, $(a_{k_n})_{n=1}^{\infty}$ is monotone.

□

Theorem 104 (Bolzano-Weierstrass Theorem)
Let (a_n) be a bounded sequence. Then (a_n) has a convergent-subsequence.

Proof. By **??**, (a_n) has a monotone subsequence, $(a_{k_n})_{n=1}^{\infty}$. Since (a_n) is bounded, so is (a_{k_n}). By the Monotone Convergence Theorem, (a_{k_n}) is convergent. □

Proof. Let (a_n) be a bounded sequence. Then there exist A_1 and B_1 such that $A_1 \leq a_n \leq B_1$ for all n. We construct a nested sequence of intervals $[A_n, B_n]$ as follows: Let $[A_2, B_2]$ be one of the intervals $[A_1, \frac{A_1+B_1}{2}]$ or $[\frac{A_1+B_1}{2}, B_1]$ such that $\{n \in \mathbb{N} \mid a_n \in [A_2, B_2]\}$ is infinite. Similarly, let $[A_3, B_3]$ be one of the intervals $[A_2, \frac{A_2+B_2}{2}]$ or $[\frac{A_2+B_2}{2}, B_2]$ such that $\{n \in \mathbb{N} \mid a_n \in [A_3, B_3]\}$ is infinite. We continue this process, ensuring that each interval $[A_n, B_n]$ contains infinitely many terms of the sequence (a_n). We then pick a subsequence $k_1 < k_2 < \ldots < k_n$ such that $a_{k_n} \in [A_n, B_n]$. Since $B_n - A_n = \frac{B_1 - A_1}{2^{n-1}}$, the length of the intervals $[A_n, B_n]$ tends to 0 as $n \to \infty$. Since (A_n) and (B_n) are monotone and bounded, they converge to the same limit L. By the Squeeze Theorem, the subsequence (a_{k_n}) converges to L. □

14 February 10, 2025

Today is exam day so no notes for today.

15 February 14, 2025

15.1 Properly Divergent Sequences

Definition 105

Let (a_n) be a sequence. We say that (a_n) is:

- We say that a_n tends to ∞ if $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ such that $n \geq N \implies a_n > M$.
- We say that a_n tends to $-\infty$ if $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ such that $n \geq N \implies a_n < M$.

If either of these hold, we say that a_n we say that a_n is **properly divergent**.

Fact 106

(a_n) diverges if there is no real number L such that $(a_n) \to L$. Every properly divergent sequence is divergent. But not every divergent sequence is properly divergent. For example $(a_n) = (-1)^n$ is divergent but not properly divergent. Also we have another sequence $b_n = (0, 1, 0, 2, 0, 3, ...)$ is unbounded and divergent but not properly diverge

Example 107

Let $a_n = n^2 + 4n - 2$ for all $n \in \mathbb{N}$. Then a_n is properly divergent.

Proof. Let $M \in \mathbb{R}$. We know that $a_n = n^2 + 4n - 2 \geq n - 2$. By Archimedean Property, there exists $N \in \mathbb{N}$ such that $n \geq N \implies n - 2 > M$. Hence $n \geq N \implies a_n = n^2 + 4n - 2 > M$. Therefore $a_n \to \infty$. □

Example 108

Let $a > 1$ and $b_n = a^n$. Then b_n is properly divergent.

Proof. Since $a > 1$, $a = 1 + x$ for some $x > 0$. By Bernouli's identity,

$$b_n = (1 + x)^n \geq 1 + nx > nx$$

Let $M \in \mathbb{R}$. By Archimedean Property, there exists $N \in \mathbb{N}$ such that $n \geq N \implies nx > M$. Hence $n \geq N \implies b_n = a^n > M$. Therefore $b_n \to \infty$. $\square$

Theorem 109

Let (a_n) be a sequence of positive real numbers. Then the following are equivalent:

(a) $(a_n) \to \infty$ as $n \to \infty$

(b) $\frac{1}{a_n} \to 0$ as $n \to \infty$

Theorem 110

Let (a_n) be a sequence. Then the following are equivalent:

(a) If (a_n) is increasing and not bounded above, then $(a_n) \to \infty$ as $n \to \infty$

(b) If (a_n) is decreasing and not bounded above, then $(a_n) \to -\infty$ as $n \to \infty$

Proof. Let $M \in \mathbb{R}$. Since (a_n) is not bounded below, for some $N \in \mathbb{N}$ such that $a_N < M$. Since (a_n) decreasing for any $\square$

Corollary 111 (Monotone Convergence Theorem III)

Let (a_n) be a sequence:

(a) If (a_n) is increasing, either it converges to some $L \in \mathbb{R}$ or $(a_n) \to \infty$ as $n \to \infty$

(b) If (a_n) is decreasing, either it converges to some $L \in \mathbb{R}$ or $(a_n) \to -\infty$ as $n \to \infty$

Example 112

Let a_n be a sequence such that $(a_n) \to \infty$ as $n \to \infty$. Then prove or find a counterexample to each of the following:

(a) If (b_n) is bounded above and $b_n \neq 0$ for all n, then $(\frac{a_n}{b_n}) \to \infty$ as $n \to \infty$.

(b) If (b_n) is bounded above and $b_n > 0$ for all n, then $(\frac{a_n}{b_n}) \to \infty$ as $n \to \infty$.

(b) If $(b_n) \to L$ for some $L > 0$ and $b_n \neq 0$ for all n, then $(\frac{a_n}{b_n}) \to \infty$ as $n \to \infty$.

Proof. So here, (a) is false and we will prove that by a counterexample. If $(a_n) = (n)$ and $(b_n) = (-1)$ then $(\frac{a_n}{b_n}) = (-n) \to -\infty$ as $n \to \infty$. So (a) is false.
(b) is true. Since (b_n) is bounded above and positive there exists $k \in \mathbb{R}$ such that $0 < b_n < k$ for all n. Let $M \in \mathbb{R}$ Since $a_n \to \infty$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $a_n > M_k$. Hence for $n \geq N$, $\frac{a_n}{b_n} > \frac{a_n}{k} > \frac{M_k}{k} = M$. So $(\frac{a_n}{b_n}) \to \infty$ as $n \to \infty$.
(c) is true. Since $b_n \to L$ for some $L > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies b_n > \frac{L}{2}$. Hence for $n \geq N$, $\frac{a_n}{b_n} > \frac{a_n}{\frac{L}{2}} = \frac{2a_n}{L} > \frac{2M_k}{L} = M$. So $(\frac{a_n}{b_n}) \to \infty$ as $n \to \infty$. $\square$

16 February 17, 2025

16.1 Limsup and Liminf

In general, a sequence may or may not converge. In the cases where it does not converge, it may or may not properly diverge. Even when it does not properly diverge, there is still a lot we might want to say about the long-term behavior of the sequence.

Proposition 113

Let (a_n) be a sequence. For each $n \in \mathbb{N}$, we set $A_n = \{a_m \mid m \geq n\}$.

1. If (a_n) is bounded above, then $(\sup(A_n))$ is a decreasing sequence of real numbers. Consequently $\lim_{n\to\infty}(\sup(A_n))$ is either a real number or $-\infty$.

2. If (a_n) is bounded below, then $(\inf(A_n))$ is an increasing sequence of real numbers. Consequently $\lim_{n\to\infty}(\inf(A_n))$ is either a real number or ∞.

Proof. We first prove (1). If (a_n) is bounded above, then each A_n is bounded above, and so $\sup(A_n)$ exists. We remark that if $m \leq k$, then $A_k \subseteq A_m$. Consequently, $\sup(A_m)$ is an upper bound of A_k, which implies $\sup(A_k) \leq \sup(A_m)$. This shows $(\sup(A_n))$ is a decreasing sequence. By Corollary 2.6.7, $(\sup(A_n))$ either converges, or it properly diverges to $-\infty$. (2) is proved similarly. □

Definition 114

Let (a_n) be a sequence. For each $n \in \mathbb{N}$, we set $A_n = \{a_m \mid m \geq n\}$.

1. We define the limit superior of (a_n), denoted by $\limsup_{n\to\infty} a_n$ or simply $\limsup a_n$, as follows:

   - If (a_n) is bounded above, we define

   $$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup(A_n) = \lim_{n\to\infty} \sup(\{a_m \mid m \geq n\}).$$

   - If (a_n) is not bounded above, then we set $\limsup_{n\to\infty} a_n = \infty$.

2. We define the limit inferior of (a_n), denoted by $\liminf_{n\to\infty} a_n$ or simply $\liminf a_n$, as follows:

   - If (a_n) is bounded below, we define

   $$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf(A_n) = \lim_{n\to\infty} \inf(\{a_m \mid m \geq n\}).$$

   - If (a_n) is not bounded below, then we set $\liminf_{n\to\infty} a_n = -\infty$.

Example 115

- Let $(a_n) = ((-1)^n)$. For each n, $A_n = \{-1, 1\}$, so $\sup(A_n) = 1$ and $\inf(A_n) = -1$. Therefore $\liminf_{n\to\infty} a_n = -1$ and $\limsup_{n\to\infty} a_n = 1$.
- Let $(b_n) = (0, 1, 0, 2, 0, 3, \ldots)$, i.e.

$$b_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

  Let $B_n = \{b_m \mid m \geq n\}$. Then $\inf(B_n) = 0$ and each B_n is unbounded. Therefore, $\liminf_{n\to\infty} b_n = 0$ and $\limsup_{n\to\infty} b_n = \infty$.
- Let $(c_n) = (-1/1, 1, -1/2, 2, -1/3, 3, \ldots)$, i.e.

$$c_n = \begin{cases} -1/(n+1)/2 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

  Let $C_n = \{c_m \mid m \geq n\}$. Then $(\inf C_n)$ is the sequence $(-1/1, -1/2, -1/2, -1/3, -1/3, -1/4, \ldots)$ which converges to zero. Thus $\liminf_{n\to\infty} c_n = 0$. Since (c_n) is not bounded above, $\limsup_{n\to\infty} c_n = \infty$.

Proposition 116

If (a_n) is a sequence, then $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$.

Remark 117. *In the case* $\liminf_{n\to\infty} a_n$ *or* $\limsup_{n\to\infty}$ *are equal to either* $-\infty$ *or* ∞*, we interpret* $\leq$ *so that* $-\infty \leq a \leq \infty$ *for all* $a \in \mathbb{R} \cup \{\pm\infty\}$*.*

Proof. Let A_n be as above. If (a_n) is not bounded below, then $\liminf_{n\to\infty} a_n = -\infty$, and so we are done since $-\infty \leq x$ for any $x \in \mathbb{R} \cup \{\pm\infty\}$. A similar argument holds if (a_n) is not bounded above. We thus assume (a_n) is bounded.

For each n, we have $\inf(A_n) \leq \sup(A_n)$. By Proposition 2.7.1, $\inf(A_n)$ is increasing and $\sup(A_n)$ is decreasing, and so

$$\inf(A_1) \leq \inf(A_n) \leq \sup(A_n) \leq \sup(A_1)$$

for all n. It follows that the sequences $(\inf(A_n))$ and $(\sup(A_n))$ are monotone and bounded, hence convergent by the Monotone Convergence Theorem. By Theorem 2.1.10, it follows that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$. □

Theorem 118

Let (a_n) be a sequence and let L be a value in $\mathbb{R} \cup \{\pm\infty\}$. The following are equivalent:

1. $\lim_{n\to\infty} a_n = L$;
2. $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L$.

Proof. As usual, we set $A_n = \{a_m \mid m \geq n\}$.

(1) ⇒ (2):

Suppose $\lim_{n\to\infty} a_n = L$. Let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. Hence,

$$L - \epsilon < a_n < L + \epsilon$$

This implies:

$$L - \epsilon \leq \inf(A_n) \leq \sup(A_n) \leq L + \epsilon$$

Therefore, $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L$.

(2) ⇒ (1):

Suppose $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L$. Let $\epsilon > 0$. Pick $N_1, N_2 \in \mathbb{N}$ such that:

$$\forall n \geq N_1 \quad |\sup(A_n) - L| < \epsilon$$

$$\forall n \geq N_2 \quad |\inf(A_n) - L| < \epsilon$$

Setting $N = \max(N_1, N_2)$, we see that for all $n \geq N$, since $a_n \in A_n$, we have:

$$L - \epsilon < \inf(A_n) \leq a_n \leq \sup(A_n) < L + \epsilon$$

This implies $a_n \to L$.

Case $L = -\infty$: This is similar to the case $L = \infty$. □

17 February 21, 2025

17.1 Series

Definition 119

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. For $n \in \mathbb{N}$, the nth partial sum is the finite sum

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

Definition 120

By the series $\sum_{i=1}^{\infty} a_i$ or just $\sum a_i$, we mean the sequence of partial sums (s_n).

Remark 121. *In the previous section, we usually considered sequences $(a_n)_{n=1}^{\infty}$ whose first term is a_1, second term is a_2, etc. However, it will be useful to consider sequences $(a_n)_{n=K}^{\infty}$ where $K \in \mathbb{Z}$, whose first term is a_K, second term is a_{K+1}, etc. The corresponding series are written as $\sum_{i=K}^{\infty} a_i$. (Usually $K = 1$ or $K = 0$.)*

Example 122

There are multiple types of series:

- **Geometric Series.** Let $x \in \mathbb{R}$, and let $(a_n)_{n=0}^{\infty} = (x^n)_{n=0}^{\infty}$ for $n \geq 0$. Then $\sum_{n=0}^{\infty} x^n$ is

$$(1, 1+x, 1+x+x^2, \dots, 1+x+x^2+\dots+x^n, \dots).$$

- **Harmonic Series.** Let $a_n = \frac{1}{n}$. Then $\sum_{n=1}^{\infty} \frac{1}{n}$ is

$$\left(1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, \dots\right).$$

- **Exponential Series.** Let $x \in \mathbb{R}$ and let $a_n = \frac{x^n}{n!}$ for $n \geq 0$. Then $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is

$$\left(1, 1+x, 1+x+\frac{x^2}{2!}, \dots\right).$$

- **Cosine Series.** Let $x \in \mathbb{R}$ and set

$$a_n = \begin{cases} \frac{x^{2m}}{(2m)!}(-1)^m & \text{if } n = 2m, \\ 0 & \text{otherwise.} \end{cases}$$

  Then $\sum_{n=0}^{\infty} a_n$ is

$$\left(1, 1, 1-\frac{x^2}{2!}, 1-\frac{x^2}{2!}, 1-\frac{x^2}{2!}+\frac{x^4}{4!}, \dots\right).$$

Definition 123

Let a_n be a sequence. We say $\sum a_n$ converges if the sequence s_n of partial sums converges.

If $a_n \to L$ as $n \to \infty$ [where $L \in \mathbb{R}$ or$L = \pm\infty$] then we write

$$\sum a_n = \sum_{n=1}^{\infty} a_n = L$$

and we say that L is the sum of the series.

Remark 124. *The partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=k}^{\infty} a_n$ differ by a constant factor (e.g. if $k > 1$, $\sum_{n=1}^{k-1} a_n$). Hence $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=k}^{\infty} a_n$ converges. However, the sums of those series must be different.*

Proposition 125 (The Term Test)

If $\sum a_n$ converges, then (a_n) converges to 0.

Proof. Suppose $\sum a_n$ converges and s_n is the sequence of partial sums. For each $n \in \mathbb{N}$, $a_{n+1} = s_{n+1} - s_n$. Since s_n converges to some $L \in \mathbb{R}$, hence does the tail s_{n+1}. By the Algebra of Limits, we have a_{n+1} converges to $L - L = 0$, which means that a_n converges to 0. □

Example 126

Let $x \in \mathbb{R}$ and $(a_n)_{n=0}^{\infty} = (x^n)_{n=0}^{\infty}$. The partial sum $s_n = 1 + x + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$.

Proof. Here we will consider two cases:

- If $\mid x \mid < 1$, then $x^{n+1} \to 0$, so $s_n \to \frac{1}{1-x}$ by the Algebra of Limits.
- If $\mid x \mid > 1$, then $x^{n+1} \not\to 0$, hence $\sum x^n$ does not converge by the (contrapositive) of the term test.

$\square$

Example 127 (Important for Counterexamples)

Let $a_n = \frac{1}{n}$. Then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Here we will consider the partial sum of the series.

$$s_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n}$$

We observe that:

$$\begin{aligned} s_{2n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2n-1} + \frac{1}{2n}\right) \\ &\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2n} + \cdots + \frac{1}{2n}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\ &= 1 + \frac{n}{2} \end{aligned}$$

So s_{2n} has a subsequence which is divergent and so is itself divergent. $\square$

18 March 3, 2025

18.1 Test for convergence

Theorem 128 (The Term Test)

If $\sum a_n$ converges, then $a_n \to 0$ as $n \to \infty$.

Proof. Suppose $\sum a_n$ converges and let s_n be the sequence of partial sums. For each $n \in \mathbb{N}$, we have $a_{n+1} = s_{n+1} - s_n$. Since s_n converges to some $L \in \mathbb{R}$, so does the tail s_{n+1}. By the Algebra of Limits, we have $a_{n+1} \to L - L = 0$, which means that $a_n \to 0$. $\square$

Theorem 129 (The comparison test)

Let (a_n) and b_n be sequences such that $0 \leq a_n \leq b_n$ such that for all $n \in \mathbb{N}$ we have the following

- If $\sum b_n$ converges, then $\sum a_n$ converge.
- If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Combined with some elementary examples this is a powerful method for detemrining convergence or divergence of series. The most important examples are following:

- $\sum x_n$ converges if $|x| < 1$ but diverges if $|x| \geq 1$
- $\sum \frac{1}{n}$ converges
- $\sum \frac{1}{n^2}$ diverges

Example 130

Let's look at some examples now:

- $\sum n^{\frac{-5}{2}}$
- $\sum \frac{1}{n(n+1)(2+\cos(n))}$
- $\sum \frac{x^n}{n}$ where $|x| < 1$

Theorem 131 (name of the theorem)

Let (a_n) be a sequence with $a_n \neq 0$ for all n. Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ (which basically means that the limit exists and it is L). Then the following holds:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$, then $\sum a_n$ diverges.
- If $L = 1$, then the

Proof Other Information. There are three parts of the proof that we need to prove: **1**: Pick $k \in \mathbb{R}$ such that $L < K < 1$. Since $\left|\frac{a_{n+1}}{a_n}\right| \to L$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\left|\frac{a_{n+1}}{a_n}\right| < k$. Now, by induction, for $L \geq 0$,

$$|a_{N+1}| \leq |a_N| k$$

Hence by comparison, with $\sum |a_N| < K^L$. We see that $\sum |a_{N+L}|$ converges. Hence, $\sum a_{N+1}$ converges, hence $\sum a_n$ converges absolutely. ☐

19 March 7, 2025

Theorem 132

19.1 Power Series

Definition 133 (Power Series)

By a **power series** mean a series of the form $\sum a_n x^n$, where $x \in \mathbb{R}$ and a_n is a sequence. We think of x as the variable and a_n is the constant.

Example 134

The following series are the exmaples of a power series:

- Geoemetric Series $\sum x_n$
- Exponential Series $\sum \frac{x^n}{n}$
- Sine or Cosine Series

Definition 135 (Radius of Convergence)

Suppose $\sum a_n x^n$ is a power series. We define $S = \{x \in \mathbb{R} \mid \sum a_n x^n \text{ converges}\}$. The **radius of convergence** R is defined as $R = \sup\{|x| \mid x \in S\}$.

Lemma 136

Suppose that $\sum a_n x^n$ is a power series and for some $x_0 \in \mathbb{R}, \sum a_n {x_0}^n$ converges. If $x \in \mathbb{R}$ such that $\mid x \mid<\mid x_0 \mid$ then $\sum a_n x^n$ converges absolutely.

Proof. Since $\sum a_n {x_0}^n$ converges, we have $a_n {x_0}^n \to 0$. Therefore, we have $\exists M \in \mathbb{R}$ such that $\mid a_n {x_0}^n \mid < M$ for all $n \in \mathbb{N}$. Now suppose that $x \in \mathbb{R}$ such that $\mid x \mid<\mid x_0 \mid$. Then $\mid x_0 \mid \neq 0$. Then the following holds:

$$|a_n x^n| = |a_n x_0^n| \left| \left(\frac{x}{x_0} \right)^n \right| < M \left| \left(\frac{x}{x_0} \right)^n \right|$$

Since $\left| \frac{x}{x_0} \right| < 1$, we can apply the comparison test with the geometric series $\sum \left(\frac{x}{x_0} \right)^n$, which converges. Therefore, $\sum a_n x^n$ converges absolutely. □

Theorem 137

Suppose that $a_n x^n$ is a power series with radius of convergence R. Then:

- If $|x| < R$, then the series converges absolutely.
- If $|x| > R$, then the series diverges.

Remark 138. *If* $|x| = R, \sum a_n x^n$ *may or may not diverge, for example* $\sum \frac{x^n}{n}$ *converges for* $|x| = 1$ *but* $\sum x^n$ *diverges for* $|x| = 1$.

Proof. If $|x| > R$, then $x \notin S$, so $\sum a_n x^n$ diverges. If $|x| < R$, by approximation property, then $\exists x_0 \in \mathbb{R}$ such that $|x| < |x_0| < R$. By Lemma **??**, $\sum a_n x^n$ converges absolutely. ☐

Proposition 139

Let $\sum a_n x^n$ be a power series and let $L = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. Then the radius of convergence of the series is given by $R = \frac{1}{L}$ or $R = 0$ if $L = \infty$.

Proof. Let $b_n = a_n x^n$. Then $\limsup |b_n|^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{n}} |x|$. By the root test, $\sum b_n$ converges if $\limsup |b_n|^{\frac{1}{n}} < 1$. Therefore, $|x| < \frac{1}{L}$ if $L < \infty$. If $L = \infty$, then $|x| < 0$. Therefore, the radius of convergence is given by $R = \frac{1}{L}$ or $R = 0$ if $L = \infty$. ☐

Example 140

Now we will look at some examples:

- $\sum x^n$ converges if $|x| < 1$, hence $R = 1$.
- $\sum \frac{x^n}{n}$ converges if $|x| < 1$, hence $R = 1$.
- $\sum \frac{x^n}{n!}$ converges if $|x| < 1$, hence $R = 1$.

20 March 17, 2023

Limits of functions and continous functions

We'll study functions $f : E \to R$ where $E \subseteq \mathbb{R}$.

20.1 Limits of functions

Definition 141

Let $E \subseteq \mathbb{R}$. We say $a \in \mathbb{R}$ is a **limit point** of E. We can also call limit point as accumulation point, or cluster point. If for all $\delta > 0, \exists Z \in R$ such that $0 < |z - a| < \delta$

Definition 142

A point of E that is not a limit point of E is an **isolated point** of E.

Theorem 143

Let $E \subseteq R$ and $a \in \mathbb{R}$. Then the following are equivalent:

(i) a is a limit point of E.

(ii) There exists a sequence (a_n) such that $(a_n) \in E \setminus \{a_n\}$ for all n and $\lim a_n = a$.

Proof. **(ii) ⇒ (i):**
Let (a_n) be as in (ii). Let $\delta > 0$. Since $\lim a_n = a$, $\exists N \in \mathbb{N}$ such that $|a_N - a| < \delta$. By assumption, we have $a_N \in E \setminus \{a_N\}$, so

$$0 < |a_N - a| < a$$

Thus a is a limit point.
(i) ⇒ (ii):
For each $n \in \mathbb{N}$, since a is a limit point of $E, \exists a_N \in E$ such that $0 < |a_N - a| < \frac{1}{n}$. The sequence a_n is contained in $E \in \setminus\{a\}$. To show that $\lim a_n = a$, let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then $\forall n \geq N$, $|a_n - a| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$. □

Theorem 144

Let $a, b \in \mathbb{R}$ with $a < b$. Let E be one of $[a, b], (a, b), [a, b), (a, b]$. Then $p \in \mathbb{R}$ is a limit point of E if and only if $p \in [a, b]$.

Proof. There are three cases that we need to consider in this case, $p < a, p \in [a, b], p > b$

Case 1: $p < a$

Let

Case 2: $p < a, p \in [a, b], p > b$

Case 3: $p < a, p \in [a, b], p > b$

□

21 March 28, 2025

Proposition 145

Let $E \subseteq \mathbb{R}$ and $a \in \mathbb{R}$ is a limit point of E. Suppose, we have functions $m, f, M : E \to \mathbb{R}$ and $\exists \delta > 0$ such that $\forall x \in E$ such that $0 < |x - a| < \delta$, we have:

$$m(x) \leq f(x) \leq M(x)$$

Suppose for some $y \in \mathbb{R}$ we have $\lim_{x \to a} m(x) = y$ and $\lim_{x \to a} M(x) = y$. We want to define $\lim_{x \to a} f(x) = y$ when either a or y are $\pm\infty$.

Definition 146

If $E \subseteq \mathbb{R}$, $a \in \mathbb{R}$, $a \in \mathbb{R}$ is a limit point of E and $f : E \to \mathbb{R}$, we write $\lim_{x\to a} f(x) = \infty$ if the following holds:

$$\forall M \in \mathbb{R} \exists \delta > 0$$

such that $\forall x \in E$ such that

$$0 < |x - a| < \delta$$

then $f(x) > M$

Definition 147

Suppose $E \subseteq \mathbb{R}$ is not odd above and $y \in \mathbb{R}$. Let $f : E \to \mathbb{R}$. Then $\lim_{xto\infty} f(x) = y$ if the following holds

$$\forall \epsilon > 0, \exists \delta > 0$$

such that $\forall x \in E$ such that

$$0 < |x - a| < \delta$$

then $|f(x) - y| < \epsilon$.

Remark 148. *Informally, the condition E is not odd above means that '∞ is a limit point of E'*

21.1 Continuity of functions

Example 149

We define $f : \mathbb{R} \to \mathbb{R}$ be as follows: $\begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ Then $\lim_{x\to 0} f(x) = 0$ but $f(0) = 1$

Definition 150

Let $f : E \to \mathbb{R}$ where $E \subseteq \mathbb{R}$ and $a \in E$. We say f is **continous** if the following holds:

$$\forall \epsilon > 0 \exists \delta > 0$$

such that $\forall x \in E \qquad |x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

Remark 151. *Recall, that $x \in E$ is an isolated point of E it is not a limit point of E.*

Proposition 152

$f : E \to \mathbb{R}$ is continous at any isolated point $a \in E$.

Proof. Since $a \in E$ is not a limit point of E, there exists $\forall \delta > 0$ such that $x \in E$ such that $|x = a| < \delta$ ☐

Example 153

Any function $f : \mathbb{N} \to \mathbb{R}$ is continous at all $n \in \mathbb{N}$

Proposition 154

Let $f : E \to \mathbb{R}$ and let $a \in E$ be a limit point of E. Then f is continous at a if and only if $\lim_{x \to a} f(x)$ exists and is equal to $f(a)$.

The proof of this follows from the definitions.

Theorem 155 (Sequential criterion for continouity)

Let $f : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}$ and $a \in E$. The following are equivalent:

- $f(x)$ is continous at a
- For any sequence (a_n) in E such that $\lim_{n \to \infty} a_n = a$. The proof of this is very simliar to