Math 54: Topology

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Introduction

Professor Vladimir Chernov (Tchernov) is the course instructor for this quarter. Office hours, class materials, lecture notes will be available on Canvas. There will be weekly homework which is worth 20% of the final grade, a midterm (40%), and a final exam (40%).

For this course, we will use *Topology* by James R. Munkres (2nd edition). The book is available for purchase online or at the Dartmouth bookstore. You can also access it here.

We will cover the first four chapters of the book, which are as follows:

- **Weeks 1, 2:** Chapter 1 Set Theory and Logic
- **Weeks 3, 4, 5:** Chapter 2 Topological Spaces and Continuous Functions
- **Weeks 6, 7:** Chapter 3 Connectedness and Compactness
- **Weeks 8, 9:** Chapter 4 Countability and Separation Axioms

Contents

1 Set Theory and Logic

1 Fundamental Concepts

We started the class with discussing some basic notation of set theory. For example, $\in, \subset, \cup, \cap, \emptyset$. Here, are usecases of that. For example,

- $a \in A$ means that a is an element of A.
- $A \subset B$ implies that set A is a subset of set B.
- $B = \{x \mid x \text{ is an even integer}\}$ is notation for the set all even integers
- $A \cap B = \{x \mid x \in A \text{ or } x \in B\}$

Example 1.1
If $x^2 < 0 \implies x = 23$. The contrapositive of that would be $x \neq 23 \leftarrow x^2 \geq 0$. The statement and the contrapositive both are true.

Theorem 1.2
Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. We will prove by showing that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. Let's start by showing $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Suppose, we have $x \in A \cap (B \cup C)$. That means that $x \in A$ and $x \in (B \cup C)$. So that means that $x \in B$ or $x \in C$. Combining them, we get $x \in A$

Now we will prove the other way. Let's start by considering both cases possibe.

- Case α: $x \in A \cap B$
- Case β: $x \in A \cap C$

□

Definition (Power of Set)
The set of all subsets of a set A is called the **power set** of A and is denoted by $\mathcal{P}(A)$.

Definition (Binary Operation)
A **binary operation** from a set A is function f mapping $A \times A$ into A.

2 Functions

3 Relations

Definition (Relation)

A **relation** on a set A is a subset of the cartesian product $A \times A$.

We denote xCy to say that $(x, y) \in C$, and we read this as x is in the relation C to y.

Example 3.1

P is the set of all people $D \subset P \times P$ is given by the equation $D = \{(x, y) \mid x \text{ is a descendant of } y\}$.

Definition (Equivalence Relation)

A relation C on a set A is an **equivalence relation** if it is

- Reflexive: $x \sim x, \forall x \in A$
- Symmetric: If $x \sim y$, then $y \sim x$
- Transitive: If $x \sim y$ and $y \sim z$, then $x \sim z$

Example 3.2

Being blood relative is an equivalence relation if you think that every person is a relative of themselves. Being descendant is not an equivalence relatoin though.

Fact 3.3

For equivalence relation C, we generally write $x \sim y$ instead of xCy. Given an equivalence relation $\sim$, an equivalence class is determined by x is denoted by $[x]$ where, $[x] = \{y \in A \mid y \sim x\}$.

Lemma 3.4

Two equivalent classes are either disjoint or equal.

Proof. Let $[x]$ and $[\tilde{x}]$ are two equivalence classes. Suppose we have $y \in [x]$, and $y \in [\tilde{x}]$. Therefore $y \sim x$, and $y \sim \tilde{x}$. Using symmetry, we can write $x \sim y$. Now, we have $x \sim y$, and $y \sim \tilde{x}$. Using transitivity, we can write $x \sim \tilde{x}$. Therefore, we can write $[x] \sim [\tilde{x}]$. Therefore, if we have $[x] \cap [\tilde{x}] \neq \emptyset$, $[x] = [\tilde{x}]$. ☐

Definition (Partition)

A **partition** of a set A is a collection of disjoint non-empty subsets of A whose union is A.

Definition (Order Relation)

An **order relation** is a relation $<$ on a set A such that

- Comparability: For every $x, y \in A$ with $x \neq y$, either $x < y$ or $y < x$.
- Nonreflexivity: For no $x \in A$ does the relation $x < x$ hold.
- Transitivity: If $x < y$ and $y < z$, then $x < z$.

As the tilde, $\sim$, for equivalence relations, we generally write $x < y$ instead of $x < y$ for order relations.

Definition (Open Interval, Immediate Predecessor and Successor)

If X is a set and $<$ is an order relation on X, and if $a < b$, then b is called an **immediate successor** of a if there does not exist $c \in X$ such that $a < c < b$. Similarly, a is called an **immediate predecessor** of b if there does not exist $c \in X$ such that $a < c < b$. The **open interval** with endpoints a and b is the set $(a, b) = \{x \in X \mid a < x < b\}$.

4 The Integers and the Real Numbers

Definition (Binary Relation)

A **binary relation** on a set A is a subset of the cartesian product $A \times A$.

Definition (Function)

Function f from a set A to a set B is a relation from A to B such that for each $a \in A$, there is a unique $b \in B$ such that $(a, b) \in f$. We write $f : A \to B$. If $(a, b) \in f$, we write $f(a) = b$.

We assume that we have two binary operations $+$ and $\cdot$ on both A and B, and we have an order relation $<$ on both A and B. Then the following properties hold:

Lemma 4.1

Let $f : A \to B$. If there exist functions $g : B \to A$ and $h : B \to A$ such that $g \circ f = a \forall a$ and $f \circ h = a \forall a$, then f is bijective and $g = h = f^{-1}$.

5 Cartesian Products

6 Finite Sets

7 Countable and Uncountable Sets

8 The Principle of Recrusive Definition

9 Infinite Sets and Axiom of Choice

10 Well-Ordered Sets

2 Topological Spaces and Continuous Functions

The concept of topological spaces is a generalization of the concept of open setes, which is defined on a metric space.

12 Topological Spaces

Definition (Topology and Topological Spaces)

A **topology** on a set X is a collection τ of subsets of X satisfying the following properties:

- $\emptyset \in \tau$ and $X \in \tau$.
- The union of any collection of sets in τ is in τ.
- The intersection of any two sets in τ is in τ.

A set X together with a topology τ is called a **topological space** and is denoted by (X, τ).

Example 12.1

The collection of all subsets of a set X forms a topology on X, called the discrete topology.

Definition (Discrete and Indiscrete Topologies)

If X is any set, the collection of all subsets of X is a topology on X; it is called the **discrete topology**. The collection of consisting of only X and $\emptyset$ is a topology on X; it is called the **indiscrete topology**.

Example 12.2

Let $X = \{a, b, c\}$. The **discrete topology** on X is the collection of all subsets of X:

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}.$$

The **indiscrete topology** on X consists of only the empty set and the whole set: $\{\emptyset, X\}$.

Definition (Finer and Coarser Topologies)

Suppose that τ and τ' are two topologies on the given set X. If $\tau' \supset \tau$, then τ' is said to be **finer** than τ; if τ' properly contains τ, we say that τ' is **strictly finer** than τ. We also say that τ is **coarser** than τ', or **strictly coarser** than τ', in these two respective situations. We say that τ is **comparable** to τ' if either $\tau \subset \tau'$ or $\tau' \subset \tau$.

13 Basis for a Topology

Definition (Basis for a Topology)
If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) such that

- Every point of X belongs to at least one basis element.
- If x belongs to the intersection of two basis elements, then there exists a basis element containing x that is contained in the intersection.

Lemma 13.1
Let X be a set; let $\mathcal{B}$ be a basis for a topology on X. Then the collection τ of all unions of elements of $\mathcal{B}$ is a topology on X.

Proof. We have to prove that τ is a topology on X. We proceed by verifying the three axioms of a topology. First, note that the empty set and the set X itself can be written as unions of basis elements (where the empty union is $\emptyset$ and the union of all basis elements is X), so both $\emptyset$ and X are in τ. Second, any union of sets from τ is just a union of unions of basis elements, which is again a union of basis elements; thus, τ is closed under arbitrary unions. Third, for any two sets $U, V \in \tau$, each is a union of basis elements. The intersection $U \cap V$ can be written as the union of all intersections of basis elements from U and V; by the property of the basis, the intersection of two basis elements is a union of basis elements, so $U \cap V$ is also a union of basis elements and hence in τ. Therefore, τ is a topology on X. $\square$

Lemma 13.2
Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open subsets of X such that for each open set U in X, there exists a subcollection $\mathcal{B}_U \subset \mathcal{C}$ such that $U = \bigcup \mathcal{B}_U$. Then the collection $\mathcal{B} = \bigcup_{U \in \mathcal{C}} \mathcal{B}_U$ is a basis for the topology on X.

Proof. We have to prove that $\mathcal{B}$ is a basis for the topology on X. We proceed by verifying the two properties of a basis. First, since each $U \in \mathcal{C}$ is open, it is a union of basis elements from $\mathcal{C}$, so every point of X belongs to at least one basis element. Second, if x belongs to the intersection of two basis elements B_1 and B_2 from $\mathcal{C}$, then x belongs to some open set $U \in \mathcal{C}$ containing x, and both B_1 and B_2 are subsets of U. Therefore, there exists a basis element B containing x that is contained in $B_1 \cap B_2$; since B is a subset of U, it is also a subset of $B_1 \cap B_2$. Therefore, $\mathcal{B}$ satisfies the two properties of a basis. $\square$

Lemma 13.3
Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies τ and τ'. Then the following are equivalent:

- τ' is finer than τ.
- Every $x \in X$ and each basis element of $B \in \mathcal{B}$ containing x, there exists a basis element of $B' \in \mathcal{B}'$ containing x such that $x \in B' \subset B$.

Proof. (2) $\implies$ (1): Suppose that τ' is finer than τ. Then every open set in τ' is also open in τ. Therefore, every basis element of τ' is also a basis element of τ. Therefore, $\mathcal{B}'$ is a basis for the topology τ'. So, we have proved that τ' is finer than τ.

(1) $\implies$ (2): Suppose that τ' is finer than τ. Then every open set in τ' is also open in τ. Therefore, every basis element of τ' is also a basis element of τ. Therefore, $\mathcal{B}'$ is a basis for the topology τ'. So, we have proved that τ' is finer than τ. (2) $\implies$ (1): Suppose that τ' is finer than τ. Then every open set in τ' is also open in τ. Therefore, every basis element of τ' is also a basis element of τ. Therefore, $\mathcal{B}'$ is a basis for the topology τ'. So, we have proved that τ' is finer than τ. □

Definition

If $\mathcal{B}$ is the collection of all open intervals in $\mathbb{R}$, then the topology generated by $\mathcal{B}$ is called the **standard topology** on the real line $\mathbb{R}$.

14 The Order Topology

15 The Product Topology on X x Y

16 The Subspace Topology

17 Closed Sets and Limit Points

18 Continuous Functions

19 The Product Topology

20 The Metric Topology

21 The Metric Topology (continued)

22 The Quotient Topology

3 Connectedness and Compactness

23 Connected Spaces

Definition (Separation and Connectedness)
Let X be a topological space. A **separation** of X is a pair U, V of disjoint non-empty open sets whose union is X. The space X is said to be **connected** if there does not exist a separation of X.

We can consider a few examples of connected and disconnected spaces such as

- $\mathbb{R} \setminus \{0\}$ is disconnected.
- The disjoint union of the two closed disks $\mathbb{D}_1$ and $\mathbb{D}_2$ in $\mathbb{R}^2$ is disconnected.
- $\mathbb{Q}^2$ is disconnected in $\mathbb{R}^2$.

Now we will show example of a connected space

- Intervals are connected.
- Open (closed) disks are connected.
- $\mathbb{R}^n$ is connected.

Lemma 23.1
If Y is a subspace of X, a separation of Y is a pair of disjoint non-empty sets A, B whose union is Y, neither of which contains a limit point of the other. The subspace Y is connected if there is no separation of Y.

Proof. Suppose, we have Y as a subspace of X. We have to prove that if Y is connected, then there is no separation of Y. Suppose, we have A, B as a separation of Y. Then A and B are disjoint non-empty sets whose union is Y. Since A and B are disjoint, there exists a point $x \in A$ and $y \in B$ such that $x \neq y$. Since Y is connected, there exists a path from x to y in Y. Since Y is a subspace of X, the path is also a path in X. Therefore, X is disconnected, which is a contradiction. Therefore, Y is connected. $\square$

Lemma 23.2
If the sets C and D form a seperation of X, and if Y is a connected subspace of X, then Y must be entirely contained in either C or D.

Proof. Suppose, we have C, D as a separation of X. We have to prove that if Y is a connected subspace of X, then Y must be entirely contained in either C or D. Suppose, we have $y \in Y$. Since Y is a connected subspace of X, there exists a path from y to x in Y. Since Y is a subspace of X, the path is also a path in X. Therefore, $x \in C$ or $x \in D$. Therefore, Y must be entirely contained in either C or D. So, we have proved that if the sets C and D form a seperation of X, and if Y is a connected subspace of X, then Y must be entirely contained in either C or D. $\square$

Theorem 23.3
The union of a collection of connected subspaces of a space of X that have a point in common is connected.

Proof. Suppose, we have $X_1, \ldots, X_n$ as a collection of connected subspaces of X that have a point in common. We have to prove that the union of these subspaces is connected. Suppose, we have $U, V \subset X_1 \cup \cdots \cup X_n$, $U \cup V = X_1 \cup \cdots \cup X_n$, $U \cap V = \emptyset$. Since $X_1, \ldots, X_n$ are connected, $X_1 \cup \cdots \cup X_n$ is connected. Therefore, $X_1 \cup \cdots \cup X_n$ is disconnected, which is a contradiction. Therefore, the union of these subspaces is connected. □

Theorem 23.4
Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.

Proof. Suppose, we have A as a connected subspace of X. We have to prove that if $A \subset B \subset \overline{A}$, then B is also connected. Suppose, we have C, D as a separation of B. Then C and D are disjoint non-empty sets whose union is B. Since A is connected, A must be entirely contained in either C or D. Since $A \subset B$, B must be entirely contained in either C or D. Therefore, B is connected. □

Theorem 23.5
The image of a connected space under a continous map is connected.

Proof. Suppose, we have $f : X \to Y$ be a continous map. We have to prove that $f(X)$ is connected if X is connected. Now, we will prove using contradiction here. Suppose, we have $U, V \subset Y$, $U \cup V = Y$, $f(X) \cap U \neq \emptyset$, $f(X) \cap V \neq \emptyset$, and $f(X) \cap U \cap V = \emptyset$. Since f is continous, the pre-image of any open set of Y is also open in X. Therefore, $U, V \subset Y$, $f^{-1}(U)$ and $f^{-1}(V)$ is also open in X. Now, $f^{-1}(U) \cup f^{-1}(V) = X$, and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Therefore, X is disconnected, which is a contradiction. Thus we conclude that, $f(X)$ is connected. □

Theorem 23.6
A finite cartesian product of connected spaces is connected.

Proof. Suppose, we have $X_1, \ldots, X_n$ as a collection of connected spaces. We have to prove that the product of these spaces is connected. Suppose, we have $U, V \subset X_1 \times \cdots \times X_n$, $U \cup V = X_1 \times \cdots \times X_n$, $U \cap V = \emptyset$. Since $X_1, \ldots, X_n$ are connected, $X_1 \times \cdots \times X_n$ is connected. Therefore, $X_1 \times \cdots \times X_n$ is disconnected, which is a contradiction. Therefore, the product of these spaces is connected. □

24 Connected Subspaces of the Real Line

Definition

A simply ordered set L having more than one element is called a **linear continuum** if it the following hold:

(1) L has a the least upper bound property.

(2) If $x < y$, there exists z such that $x < z < y$.

Theorem 24.1

If L is a linear continuum in the order topology, then L is connected and so are intervals and rays in L.

Proof. Suppose, we have L as a linear continuum in the order topology. We have to prove that L is connected and so are intervals and rays in L. Suppose, we have $U, V \subset L$, $U \cup V = L$, $U \cap V = \emptyset$. Since L is a linear continuum, there exists a point $x \in L$ such that $x \in U$ and $x \in V$. Therefore, L is disconnected, which is a contradiction. Therefore, L is connected. Since L is connected, intervals and rays in L are also connected. □

Corollary 24.2

The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.

Proof. The real line $\mathbb{R}$ is a linear continuum in the order topology. Therefore, $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$. □

Theorem 24.3 (Intermediate Value Theorem)

Let $f : X \to Y$ be a continous function, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c in X such that $f(c) = r$.

Definition

Given points x and y of the space X, a **path** in X in x to y is a continous map $f : [0, 1] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is **path connected** if every pair of points of X can be joined by a path in X.

26 Compact Spaces

Definition

A collection of $\mathcal{A}$ of subsets of a space X is said to be a **cover** of X, or to be a **covering** of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called a **open covering** of X if its elements are open subsets of X.

Definition

A space X is said to be **compact** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Lemma 26.1

Let Y be a subspae of X. Then Y is compact if and only if every covering of Y by open sets in X has a finite subcollection covering Y.

Proof. Suppose, we have Y as a subspace of X. We have to prove that if Y is compact, then every covering of Y by open sets in X has a finite subcollection covering Y. Suppose, we have $\mathcal{A}$ as a covering of Y. Then $\mathcal{A}$ is a collection of open sets in X whose union is Y. Since Y is compact, there exists a finite subcollection $\mathcal{A}' \subset \mathcal{A}$ such that $\bigcup_{A \in \mathcal{A}'} A = Y$. Therefore, Y is compact. □

Theorem 26.2

Every closed subspace of a compact space is compact.

Proof. Suppose, we have Y as a closed subspace of X. We have to prove that Y is compact. Suppose, we have $\mathcal{A}$ as a covering of Y. Then $\mathcal{A}$ is a collection of open sets in X whose union is Y. Since Y is closed, $X \setminus Y$ is open. Therefore, $\mathcal{A} \cup \{X \setminus Y\}$ is a covering of X. Since X is compact, there exists a finite subcollection $\mathcal{A}' \subset \mathcal{A} \cup \{X \setminus Y\}$ such that $\bigcup_{A \in \mathcal{A}'} A = X$. Therefore, Y is compact. □

Theorem 26.3

Every compact subpsace of a Hausdorff space is closed.

Proof. Suppose, we have Y as a compact subspace of X. We have to prove that Y is closed. Suppose, we have $x \in X \setminus Y$. Then $x \in X \setminus Y$ is open. Therefore, $\mathcal{A} = \{X \setminus Y\}$ is a covering of X. Since X is compact, there exists a finite subcollection $\mathcal{A}' \subset \mathcal{A}$ such that $\bigcup_{A \in \mathcal{A}'} A = X$. Therefore, $x \in X \setminus Y$ is not in the union of the finite subcollection $\mathcal{A}'$. Therefore, $x \in X \setminus Y$ is not in Y. Therefore, Y is closed. □

Lemma 26.4 (The tube lemma)

Consider the product space $X \times Y$ where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$ at x_0, then there exists an open set U of X containing x_0 such that $x_0 \times Y \subset N$.

Proof. Suppose, we have $X \times Y$ as a product space. We have to prove that if N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$ at x_0, then there exists an open set U of X containing x_0 such that $x_0 \times Y \subset N$. Suppose, we have $\mathcal{A}$ as a covering of Y. Then $\mathcal{A}$ is a collection of open sets in X whose union is Y. Since Y is compact, there exists a finite subcollection $\mathcal{A}' \subset \mathcal{A}$ such that $\bigcup_{A \in \mathcal{A}'} A = Y$. Therefore, Y is compact. □