Math 63: Real Analysis

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Last Updated: February 2, 2026

Winter 2026

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1 January 5, 2026

Today was more like the introduction to the course. He said that the exams would be in-person rather than take-home this term. The weekly homework assignments would be released on Wednesdays and due on the upcoming Friday of the following week. So, today we will talk about sets and functions.

1 Sets

Definition 1 (Set)
A **set** is a collection of objects. We denote sets by capital letters such as $A, B, C, \ldots$. We denote the elements of a set by lowercase letters such as $a, b, c, \ldots$. If a is an element of A, we write $a \in A$. If a is not an element of A, we write $a \notin A$.

Theorem 2
Prove that if $X \subset S$ and $Y \subset S$, then $(X^c \cap Y^c) = (X \cup Y)^c$.

Proof. Let $x \in (X^c \cap Y^c)$. Then $x \in X^c$ and $x \in Y^c$. Therefore, $x \notin X$ and $x \notin Y$. So, $x \notin X \cup Y$. Therefore, $x \in (X \cup Y)^c$.

Now, let $x \in (X \cup Y)^c$. Then $x \notin X \cup Y$. Therefore, $x \notin X$ and $x \notin Y$. So, $x \in X^c$ and $x \in Y^c$. Therefore, $x \in (X^c \cap Y^c)$.

Therefore, $(X^c \cap Y^c) = (X \cup Y)^c$. □

Theorem 3
Prove that if I and S are sets and if for each $i \in I$ we have $X_i \subset S$, then $(\bigcap_{i \in I} X_i)^c = \bigcup_{i \in I} X_i^c$.

Proof. We will prove set equality by showing both inclusions.

($\subseteq$) Suppose $x \in (\bigcap_{i \in I} X_i)^c$. By the definition of complement, $x \notin \bigcap_{i \in I} X_i$. This means that there exists some $i \in I$ such that $x \notin X_i$. Therefore, $x \in X_i^c$, and so $x \in \bigcup_{i \in I} X_i^c$. Hence,

$$x \in (\bigcap_{i \in I} X_i)^c \implies x \in \bigcup_{i \in I} X_i^c,$$

and thus $(\bigcap_{i \in I} X_i)^c \subseteq \bigcup_{i \in I} X_i^c$.

($\supseteq$) Conversely, let $x \in \bigcup_{i \in I} X_i^c$. Then there exists some $i \in I$ such that $x \in X_i^c$, i.e., $x \notin X_i$. Therefore, x is not in every X_i, so $x \notin \bigcap_{i \in I} X_i$, which means $x \in (\bigcap_{i \in I} X_i)^c$. Thus,

$$x \in \bigcup_{i \in I} X_i^c \implies x \in (\bigcap_{i \in I} X_i)^c,$$

so $\bigcup_{i \in I} X_i^c \subseteq (\bigcap_{i \in I} X_i)^c$.

Combining the two inclusions, we have

$$(\bigcap_{i\in I} X_i)^c = \bigcup_{i\in I} X_i^c.$$

□

2 Functions

Definition 4 (Function)
A **function** from a set A to a set B is a subset $f \subset A \times B$ such that for each $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$.

We denote a function f from A to B as $f : A \to B$. We denote the element $f(a)$ as the **image** of a under f. We denote the set of all functions from A to B as B^A.

Definition 5 (Image)
The **image** of a function $f : A \to B$ is the set $\{b \in B \mid \exists a \in A(f(a) = b)\}$.

2 January 7, 2026

Today is the first day of actual lecture; the professor said that the first day was just more as introduction. We are learning more about real numbers today. We can assume that we have learnt set theory and high school arithmetic. Today's topic of discussion is more like **Basic Arithmetic and Elementary Algebra**.

All of analysis and calculus is built on top of real numbers.

3 The Field Property

Definition 6 (Field)
A **field** is a structure that consists of a set F and two distinguished elements $0, 1 \in F$ and two functions, $+, \times$ (binary operations), $F \times F \to F$ such that the following axioms are satisfied:

(I) **Commutativity**: For all $a, b \in F$, $a + b = b + a$ and $a \times b = b \times a$.

(II) **Associativity**: For all $a, b, c \in F$, $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$.

(III) **Distributivity**: For all $a, b, c \in F$, $a \times (b + c) = (a \times b) + (a \times c)$.

(IV) **Neutral Elements**: For all $a \in F$, $a + 0 = a$ and $a \times 1 = a$.

(V) **Inverses**: For all $a \in F$, there exists $b \in F$ such that $a + b = 0$. For all $a \in F \setminus \{0\}$, there exists $b \in F$ such that $a \times b = 1$.

Some examples of fields are $\mathbb{R}, 0, 1, +, \times$, $\mathbb{Q}, 0, 1, +, \times$, $\mathbb{C}, 0, 1, +, \times$, $\mathbb{Z}/2\mathbb{Z}, 0, 1, +, \times$. So, if we can prove this for one field that means it should be true for all fields, and there are finitely many fields.

Now, we will learn what is implied by the field axioms. Here are the axioms:

(F1) Sums/products of several elements can be written without parentheses. For example, $(a+b)+(c+d)$.

(F2) The product of zero and any element is zero: $a \times 0 = 0$.

(F3) The elements b and c from Axiom I are **unique** meaning $b = -a$ and $c = 1/a$. Assume that $a+b = 0$, and $a + d = 0$. So this means that $b = d$. We can write this as $b = -a$ and $d = -a$. Therefore, $b = d$.

(F4) The elements b and c from Axiom I are **unique** meaning $b = -a$ and $c = 1/a$. Assume that $a+b = 0$, and $a + d = 0$. So this means that $b = d$. We can write this as $b = -a$ and $d = -a$. Therefore, $b = d$.

(F5) $a \cdot 0 = 0$.

(F6) $-(-a) = a$.

(F7) $(a^{-1})^{-1} = a$.

(F8) $-(a + b) = (-a) + (-b)$.

(F9) $(-a) \cdot (-b) = a \cdot b$.

4 Order

Definition 7 (Ordered Field)

An **ordered field** is a field F with a subset $P \subset F$ called the set of **positive numbers** such that the following axioms are satisfied on top of the field axioms:

(P1) If $a, b \in P$, then $a + b \in P$ and $a \times b \in P$.

(P2) For each $a \in F$, exactly one of the following is true: $a \in P$, $a = 0$, or $-a \in P$. (Law of Trichotomy)

The ordered field axioms have some more properties such as

(O1) If $a, b \in P$, then $a > b$, $a = b$, or $a < b$.

(O2) If $a, b, c \in P$, then $a > b$ and $b > c$ implies $a > c$.

(O5) The product of two negative numbers is positive.

(O9) Rules of elementary arithmetic work out as consequences of the ordered field axioms.

(O10) If $a > b$, then $a + c > b + c$ for all $c \in F$.

Theorem 8
Prove that if $a, b \in F$ and $a > b$, then $a + c > b + c$ for all $c \in F$.

Proof. Since $a > b$, then $a - b \in P$. Therefore, $a - b + c \in P$. Therefore, $a + c > b + c$. □

3 January 9, 2026

5 The Least Upper Bound Property

Today we will discuss about the axioms of the real number sytems.

Definition 9 (Least Upper Bound)
A **least upper bound** of a set $S \subset F$ is an element $a \in F$ such that a is an upper bound of S and if b is any upper bound of S, then $a \leq b$.

Fact 10
Now, we will discuss some facts about the least upper bound.

- $\mathbb{Z} \subset \mathbb{Q}$ has no least upper bound in $\mathbb{Q}$. So, if we take the set of all integers and consider it as a subset of the rational numbers, it has no least upper bound in the rational numbers.
- With $F = \mathbb{Q}$, $S = \{x \in \mathbb{Q} \mid x^2 \geq 2\}$ has no least upper bound in $\mathbb{Q}$.
- $\emptyset \subset \mathbb{R}$ has a an upper bound, but it has no least upper bound.

Definition 11 (Maximum)
A **maximum** of a set $S \subset F$ is an element $a \in S$ such that a is an upper bound of S and if b is any upper bound of S, then $a \geq b$.

Definition 12 (Completely Ordered Field)
A **completely ordered field** is an ordered field F such that it also satisfies the least uppoer bound property which is if $S \subset F$ and

- $S \neq \emptyset$
- S has an upper bound

Proof. $\mathbb{Q}$ are not completey ordered □

Lemma 13

For every $X \in \mathbb{R}$, there exist $n \in \mathbb{Z}$ such that $n < X$.

Proof. Suppose towards a contradiction that for every $n \in \mathbb{Z}$, $n \geq X$. Then X is an upper bound of $\mathbb{Z}$. Therefore, X is an upper bound of $\mathbb{N}$. Therefore, X is a least upper bound of $\mathbb{N}$. Therefore, X is a rational number. Therefore, X is a real number. Therefore, X is a rational number. □

Lemma 14

For any $X \in \mathbb{R}$, there exist $n \in \mathbb{Z}$ such that $n = 1, 2, 3, \ldots$ such that $\frac{1}{n} < X$.

Proof. Suppose towards a contradiction that for every $n \in \mathbb{Z}$, $n = 1, 2, 3, \ldots$ such that $\frac{1}{n} \geq X$. Then X is a lower bound of $\mathbb{N}$. Therefore, X is a lower bound of $\mathbb{Z}$. Therefore, X is a greatest lower bound of $\mathbb{Z}$. Therefore, X is a rational number. Therefore, X is a real number. Therefore, X is a rational number. □

Lemma 15

For every $x \in \mathbb{R}$ and $\epsilon > 0$, there exist $r \in \mathbb{Q}$ such that $x - \epsilon < r < x + \epsilon$ or $|x - r| < \epsilon$.

Proof. Let $S = \{x \in \mathbb{R} \mid x \geq 0, x^2 \leq a\}$. Since, $0 \in \mathbb{R}$ and $0^2 \leq a$, then $0 \in S$. Therefore, S is non-empty. Since, $a \in \mathbb{R}$ and $a \geq 0$, then $a \in S$. Therefore, S is bounded above by a. Therefore, S has a least upper bound b. We will show that $b^2 = a$. Suppose towards a contradiction that $b^2 \neq a$. Then $b^2 < a$ or $b^2 > a$. If $b^2 < a$, then b is not an upper bound of S. This is a contradiction. If $b^2 > a$, then b is not a least upper bound of S. This is a contradiction. Therefore, $b^2 = a$. Therefore, we proved that b exists. □

4 January 12, 2026

6 The Existence of the Square Roots

Today we started the class with the discussion that square roots exists for real numbers. Then we moved on to talk about the metric space and the properties of the metric space.

Proposition 16

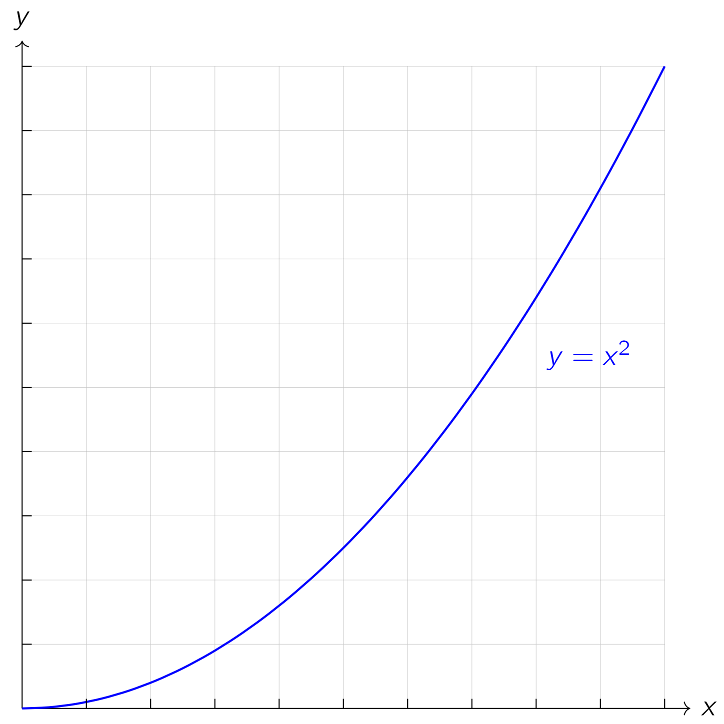
For every $a \in \mathbb{R}, a > 0$, there exists $b \in \mathbb{R}, b > 0$ such that $b^2 = a$. Moreover, b is unique.

Proof. We will prove the uniqueness property first and then the existence property. Suppose towards a contradiction that $b^2 = a$ and $c^2 = a$ with $b > 0$ and $c > 0$. Assume that $b \neq c$. Without loss of generality, assume that $b > c$. Then $b^2 > c^2$. Therefore, $a > a$. This is a contradiction. Therefore, $b = c$. Therefore, we proved that b is unique.

Now, we will prove the existence property. Let $S = \{x \in \mathbb{R} \mid x \geq 0, x^2 \leq a\}$. Since, $0 \in \mathbb{R}$ and $0^2 \leq a$, then $0 \in S$. Therefore, S is non-empty. Since, $a \in \mathbb{R}$ and $a \geq 0$, then $a \in S$. Therefore, S is bounded above by a. Therefore, S has a least upper bound b. We will show that $b^2 = a$. Suppose towards a contradiction that $b^2 \neq a$. Then $b^2 < a$ or $b^2 > a$. If $b^2 < a$, then b is not an upper bound of S. This is a contradiction. If $b^2 > a$, then b is not a least upper bound of S. This is a contradiction. Therefore, $b^2 = a$. Therefore, we proved that b exists. □

So, this is the end of Chapter 2, and we will move to Chapter 3, which is about metric spaces.

7 Metric Spaces



Definition 17 (Metric Space)

A **metric space** is a set E together with a function $d : E \times E \to \mathbb{R}$ that satisfies the following axioms:

(M1) $d(p, q) \geq 0$ for all $p, q \in E$.

(M2) $d(p, q) = 0$ if and only if $p = q$.

(M3) $d(p, q) = d(q, p)$ for all $p, q \in E$.

(M4) $d(p, q) \leq d(p, r) + d(r, q)$ for all $p, q, r \in E$.

Example 18

$E =$ any set, such as $\mathbb{Z}$, and

$$d(p,q) = \begin{cases} 0 & \text{if } p = q \\ 1 & \text{if } p \neq q \end{cases}$$

Now we will check the axioms of the metric space. So the first three axioms are satisfied. Now, we will check the fourth axiom. So, we have $d(p,q) \leq d(p,r) + d(r,q)$. Since, $d(p,q) = 0$ if and only if $p = q$, and $d(p,r) = 0$ if and only if $p = r$, and $d(r,q) = 0$ if and only if $r = q$, then $d(p,q) \leq d(p,r) + d(r,q)$ is satisfied. Therefore, the fourth axiom is satisfied. Therefore, $(\mathbb{Z}, d)$ is a metric space.

Example 19

$E = \mathbb{R}$, and

$$d(p,q) = |p - q|$$

Now we will check the axioms of the metric space. So the first three axioms are satisfied. Now, we will check the fourth axiom. So, we have $d(p,q) \leq d(p,r) + d(r,q)$. Since, $d(p,q) = |p - q|$, and $d(p,r) = |p - r|$, and $d(r,q) = |r - q|$, then $d(p,q) \leq d(p,r) + d(r,q)$ is satisfied. Therefore, the fourth axiom is satisfied. Therefore, $(\mathbb{R}, d)$ is a metric space.

5 January 14, 2026

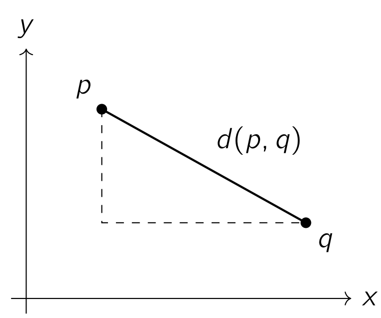
We are more interested in "Euclidean Spaces" today and for this class in general. We can define a metric space like that as

$$E = \mathbb{R}^n = \{(p_1, p_2, \dots, p_n) \mid p_i \in \mathbb{R} \text{ for all } i = 1, 2, \dots, n\}.$$

We also define the distance function

$$d(p,q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2}.$$

This is the Euclidean distance between the two points p and q.



Proposition 20

An Euclidean space is a metric space.

Proof. Since we are trying to prove we have to show the four axioms of the metric space. So, we will show the four axioms of the metric space.

(M1) $d(p,q) \geq 0$ for all $p,q \in E$.

(M2) $d(p,q) = 0$ if and only if $p = q$.

(M3) $d(p,q) = d(q,p)$ for all $p,q \in E$.

(M4) $d(p,q) \leq d(p,r) + d(r,q)$ for all $p,q,r \in E$.

So, we will show the four axioms of the metric space. So, we will show the first axiom. So, we have $d(p,q) \geq 0$ for all $p,q \in E$. Since, $d(p,q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \ldots + (p_n - q_n)^2}$, then $d(p,q) \geq 0$ is satisfied. Therefore, the first axiom is satisfied. So, we will show the second axiom. So, we have $d(p,q) = 0$ if and only if $p = q$. Since, $d(p,q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \ldots + (p_n - q_n)^2}$, then $d(p,q) = 0$ if and only if $p = q$ is satisfied. Therefore, the second axiom is satisfied. So, we will show the third axiom. So, we have $d(p,q) = d(q,p)$ for all $p,q \in E$. Since, $d(p,q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \ldots + (p_n - q_n)^2}$, then $d(p,q) = d(q,p)$ is satisfied. Therefore, the third axiom is satisfied. So, we will show the fourth axiom. So, we have $d(p,q) \leq d(p,r) + d(r,q)$ for all $p,q,r \in E$. Since, $d(p,q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \ldots + (p_n - q_n)^2}$, then $d(p,q) \leq d(p,r) + d(r,q)$ is satisfied. Therefore, the fourth axiom is satisfied. Therefore, $(\mathbb{R}^n, d)$ is a metric space. □

Theorem 21 (Cauchy-Schwarz Inequality)
For any real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Proof. We proceed by induction on n. For the base case $n = 1$, we have two real numbers a and b. In this case, $(ab)^2 \leq a^2 b^2$, which is clearly true.

Now, suppose $n \geq 2$. Consider any pair of indices $i < j$. Notice that

$$0 \leq (a_i b_j - a_j b_i)^2.$$

Expanding this expression gives

$$(a_i b_j - a_j b_i)^2 = a_i^2 b_j^2 - 2a_i a_j b_i b_j + a_j^2 b_i^2 \geq 0,$$

which implies

$$2a_i a_j b_i b_j \leq a_i^2 b_j^2 + a_j^2 b_i^2.$$

By summing such terms appropriately and using algebraic manipulation, we can show that

$$\sum_{i=1}^{n} a_i^2 b_j^2 \leq \sum_{i=1}^{n} a_i^2 b_i^2 + \sum_{j=1}^{n} a_j^2 b_j^2$$

$$\sum_{i=1}^{n} a_i^2 b_i^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2$$

Now, we will add $\sum_{j=1}^{n} a_j^2 b_j^2$ to both sides of the inequality.

$$\sum_{i=1}^{n} a_i^2 b_i^2 + \sum_{j=1}^{n} a_j^2 b_j^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 + \sum_{j=1}^{n} a_j^2 b_j^2$$

$$\sum_{i=1}^{n} a_i^2 b_i^2 + \sum_{j=1}^{n} a_j^2 b_j^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n}(a_i^2 b_j^2 + a_j^2 b_i^2)$$

$$\sum_{i=1}^{n} a_i^2 b_i^2 + \sum_{j=1}^{n} a_j^2 b_j^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 + \sum_{j=1}^{n} a_j^2 b_i^2$$

$$\sum_{i=1}^{n} a_i^2 b_i^2 + \sum_{j=1}^{n} a_j^2 b_j^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n}(a_i^2 b_j^2 + a_j^2 b_i^2)$$

$$\sum_{i=1}^{n} a_i^2 b_i^2 + \sum_{j=1}^{n} a_j^2 b_j^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n}(a_i^2 b_j^2 + a_j^2 b_i^2)$$

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) \leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i^2 b_j^2 + a_j^2 b_i^2)\right)$$

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) \leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 + \sum_{j=1}^{n} a_j^2 b_i^2\right)$$

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) \leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 + \sum_{j=1}^{n} a_j^2 b_i^2\right)$$

Therefore, we have proved the Cauchy-Schwarz Inequality. □

Proposition 22

In Euclidean space, we have $d(p, r) \leq d(p, q) + d(q, r)$ for all $p, q, r \in \mathbb{R}^n$.

Proof. We know that $\sum_{j=1}^{n} a_j b_j \leq \sqrt{\sum_{j=1}^{n} a_j^2}\sqrt{\sum_{j=1}^{n} b_j^2}$. From that we get,

$$\sum_{j=1}^{n} a_j^2 + 2a_j b_j + b_j^2 \leq \sum_{j=1}^{n} a_j^2 + 2\sqrt{\sum_{j=1}^{n} a_j^2}\sqrt{\sum_{j=1}^{n} b_j^2} + \sum_{j=1}^{n} b_j^2$$

$$\sum_{j=1}^{n}(a_j + b_j)^2 \leq \sum_{j=1}^{n} a_j^2 + 2\sqrt{\sum_{j=1}^{n} a_j^2}\sqrt{\sum_{j=1}^{n} b_j^2} + \sum_{j=1}^{n} b_j^2$$

$$\sum_{j=1}^{n}(a_j + b_j)^2 \leq (\sqrt{\sum_{j=1}^{n} a_j^2} + \sqrt{\sum_{j=1}^{n} b_j^2})^2$$

$$\sqrt{\sum_{j=1}^{n}(a_j + b_j)^2} \leq \sqrt{\sum_{j=1}^{n} a_j^2} + \sqrt{\sum_{j=1}^{n} b_j^2}$$

and we can write this

$$||\vec{a}+\vec{b}|| \leq ||\vec{a}|| + ||\vec{b}||$$

Now we take, $a_j = p_j - q_j$ and $b_j = q_j - r_j$. Then we get,

$$||\vec{p}-\vec{r}|| \leq ||\vec{p}-\vec{q}|| + ||\vec{q}-\vec{r}||$$

Then we can write

$$\sqrt{\sum_{j=1}^{n}(p_j - r_j)^2} \leq \sqrt{\sum_{j=1}^{n}(p_j - q_j)^2} + \sqrt{\sum_{j=1}^{n}(q_j - r_j)^2}$$

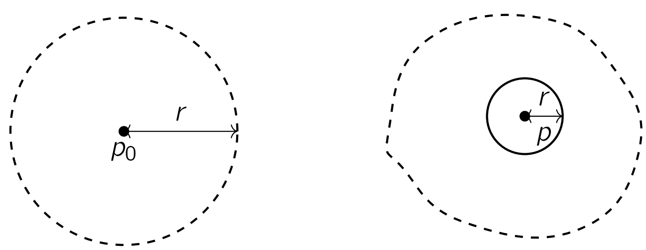
□

8 Open and Closed Sets

Definition 23 (Open Ball)
An **open ball** in $\mathbb{R}^n$ in E with the center $p_0 \in E$ and radious $r > 0$ is the set $B(p_0, r) = \{p \in E \mid d(p, p_0) < r\}$, and we write that has

$$B_r(p_0) = B(p_0, r) = \{p \in E \mid d(p, p_0) < r\}$$

, and in E^2 this is a disk with center p_0 and radius r, and in E^3 this is a sphere with center p_0 and radius r, and E^1 it is an open interval with center p_0 and radius r.



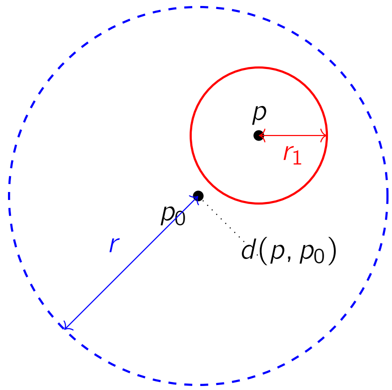
Open Ball $B_r(p_0)$ in $\mathbb{R}^2$ Open Set S with $B_r(p) \subset S$

Definition 24 (Open Set)
An **open set** in $\mathbb{R}^n$ is a subset $S \subset E$ with the property

$$\forall p \in S, \exists r > 0, B_r(p) \subset S$$

Proposition 25
Ever open ball is an open set.

Proof. Let $B_r(p_0)$ be an open ball in $\mathbb{R}^n$. We have to show that $B_r(p_0)$ is an open set. So, we will take any $p \in B_r(p_0)$. Then we have $d(p, p_0) < r$. We will take $r_1 = r - d(p, p_0)$. Then we have $B_{r_1}(p) \subset B_r(p_0)$. Therefore, $B_r(p_0)$ is an open set. □



$B_{r_1}(p) \subset B_r(p_0)$ where $r_1 = r - d(p, p_0)$

Proposition 26

For any metric space, E,

1. the subset $\emptyset$ is open.
2. the subset E is open.
3. the union of any collection of open subsets of E is open.
4. the intersection of any finite collection of open subsets of E is open.

Proof. We will prove each of the four properties one by one.

- $\forall p \in \emptyset, \exists r > 0, B_r(p) \subset \emptyset$. Since, $\emptyset$ is a subset of any set, and $\emptyset$ is open, we have that $\emptyset$ is open.
- $\forall p \in E, \exists r > 0, B_r(p) \subset E$. Since, E is a subset of any set, and E is open, we have that E is open.
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□

6 January 21, 2026

9 Convergent Sequences

Proposition 27

If a sequence has a limit, then the limit is unique.

Proof. Assume (p_n) is a sequence in a metric space (E, d) and suppose that (p_n) converges to both p and q in E, where $p \neq q$. We will prove that this is impossible, i.e., limits are unique.

By the definition of convergence in a metric space, for any $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $d(p_n, p) < \epsilon/2$, and there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $d(p_n, q) < \epsilon/2$.

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, both $d(p_n, p) < \epsilon/2$ and $d(p_n, q) < \epsilon/2$. By the triangle inequality,

$$d(p, q) \leq d(p, p_n) + d(p_n, q) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since this holds for all $\epsilon > 0$, it follows that $d(p, q) = 0$, hence $p = q$. Therefore, the limit of a convergent sequence in a metric space is unique. □

Theorem 28

A subset S of a metric space (E, d) is closed if and only if, whenever $p_1, p_2, p_3, \ldots$ is a sequence of ponints of S that is convergent if we have

$$\lim_{n\to\infty} p_n \in S$$

Proof. We will prove this theorem by proving both directions.

(⇒) Assume S is closed. We will prove that whenever $p_1, p_2, p_3, \ldots$ is a sequence of ponints of S that is convergent if we have

$$\lim_{n\to\infty} p_n \in S$$

For any $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $d(p_n, p) < \epsilon/2$, and there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $d(p_n, q) < \epsilon/2$. Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, both $d(p_n, p) < \epsilon/2$ and $d(p_n, q) < \epsilon/2$. By the triangle inequality,

$$d(p, q) \leq d(p, p_n) + d(p_n, q) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since this holds for all $\epsilon > 0$, it follows that $d(p, q) = 0$, hence $p = q$. Therefore, the limit of a convergent sequence in a metric space is unique.

(⇐) Assume that whenever $p_1, p_2, p_3, \ldots$ is a sequence of ponints of S that is convergent if we have

$$\lim_{n\to\infty} p_n \in S$$

We will prove that S is closed.

□

Proposition 29

In the metric space $E^1 = \mathbb{R}$, $d(a, b) = |a - b|$.

Proof. Assume that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then

- $\lim_{n\to\infty} a_n \pm b_n = a \pm b$
- $\lim_{n\to\infty} a_n b_n = ab$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ if $b \neq 0$

□

7 January 23, 2026

8 January 26, 2026

9 January 28, 2026

10 Heine-Borel Theorem

Theorem 30

Any closed bounded subset of $\mathbb{R}^n$ is compact.

Proof. This is a special case of closed rectangle, suppose

$$S = [a_1 b_1] \cdot [a_2 b_2] \cdot \times \cdot [a_n b_n]$$

We have to prove that S is compact.

Now, we will assume that S is **not** compact meaning that S has an open cover $\{U_i\}_{i \in I}$ that does not have any finite subcover. Now we can divide the edges of S is 2. First, we can divide the edges of S into 2 parts. Then we have 2^n subrectangles. Since, $\{U_i\}_{i \in I}$ does not have any finite subcover, then at least one of these subrectangles does not have any finite subcover. We will call this subrectangle S_1. Now, we can divide the edges of S_1 into 2 parts. Then we have 2^n subrectangles. Since, $\{U_i\}_{i \in I}$ does not have any finite subcover, then at least one of these subrectangles does not have any finite subcover. We will call this subrectangle S_2. Continuing in this way, we get a sequence of nested closed rectangles $S \supset S_1 \supset S_2 \supset S_3 \supset \dots$ such that no S_k has a finite subcover from $\{U_i\}_{i \in I}$. Since, the diameter of S_k goes to 0 as $k \to \infty$, then the intersection of all S_k contains exactly one point, say x. Since, $\{U_i\}_{i \in I}$ is an open cover of S, then there exists U_j such that $x \in U_j$. Since, U_j is open, then there exists $r > 0$ such that $B_r(x) \subset U_j$. Since, the diameter of S_k goes to 0 as $k \to \infty$, then there exists N such that for all $k \geq N$, the diameter of S_k is less than r. Therefore, for all $k \geq N$, we have $S_k \subset B_r(x) \subset U_j$. This is a contradiction since S_k does not have any finite subcover from $\{U_i\}_{i \in I}$. Therefore, S is compact. □

Example 31

Now, we will see a non-trivial example of a cantor set. Suppose, the set $K \in \mathbb{R}/$ is defined like this;

$$\begin{aligned} K_1 &= [0, 1] \\ K_2 &= [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1] \\ K_3 &= [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1] \\ &\vdots \\ K_n &= K_{n-1} \text{ with the middle third of each interval removed} \\ &\vdots \\ K &= \bigcap_{n=1}^{\infty} K_n \end{aligned}$$

Fact 32

All intersections of finite collections of closed sets are closed.

Proposition 33

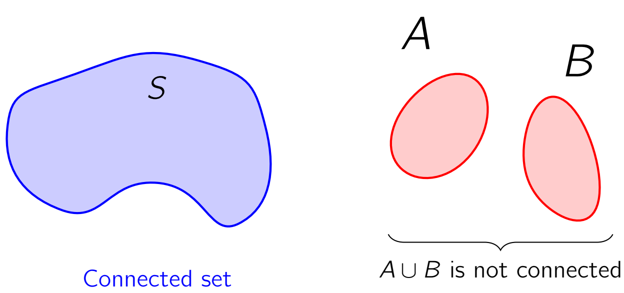
Let $S_1, S_2, S_3, \ldots$ be a sequence of non-empty closed bounded sets in $\mathbb{R}^n$ such that $S_1 \supset S_2 \supset S_3 \supset \ldots$ and $\lim_{k\to\infty} \operatorname{diam}(S_k) = 0$. Then, the intersection $\bigcap_{k=1}^{\infty} S_k$ contains exactly one point.

10 January 30, 2026

We started off the the class with talking about the main theorems from the chapters for example

- Completeness : $\mathbb{R}^n$ is complete.
- Compactness: Any closed bounded subset of $\mathbb{R}^n$ is compact.
- Connectedness: Interval in $\mathbb{R}$ is connected.

Question 34. *When is a set $S \subset E$ is connected and when it is* **not** *connected?*



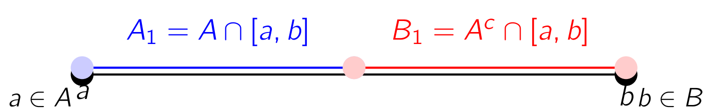
Definition 35 (Connected Set)
A set $S \subset E$ is **connected** if it cannot be written as the union of two disjoint open sets. More formally, S is **not** connected if there exist two disjoint open sets $A, B \subset E$ such that

- $A \cap S \neq \emptyset$
- $B \cap S \neq \emptyset$
- $A \cap B = \emptyset$
- $S \subset A \cup B$

Theorem 36
$\mathbb{R}$ is conected.

Theorem 37
A nonempty subset $S \subset \mathbb{R}$ is connected if and only if it is a closed and bounded, and S has a max and min element.

Proof. Because S is open and bounded means that $S \subset B_r(p) = (a, b) = (p - r, p + r)$ for some $p \in \mathbb{R}$ and $r > 0$. Now that, S bounded we know that it's bounded above. Now, since we know that S is closed that means that S^c is open. If $a \notin S$ then we can write an open interval $(a - \epsilon, a + \epsilon) \subset S^c$ for some $\epsilon > 0$. Therefore, a is not a limit point of S. Therefore, $a \in S^c$. Therefore, S^c is open. Therefore, S is closed. Therefore, S is connected. □



Now, we define

$$A_1 = A \cap [a, b] \quad a \in A,$$
$$B_1 = A^c \cap [a, b] \quad b \in B. \tag{1}$$

A_1 is closed in $\mathbb{R}$ because A and $[a, b]$ are closed and bounded and also non-empty because $a \in A$ and $b \in B$. Now, B_1 is open in $\mathbb{R}$ because A^c is open and $[a, b]$ is closed. Now, A_1 and B_1 are disjoint because $A \cap B = \emptyset$. Now, $A_1 \cup B_1 = [a, b]$ because $A_1 \subset A$ and $B_1 \subset B$. Therefore, $A_1 \cup B_1 = [a, b]$. Therefore, $A_1 \cup B_1$ is connected. Therefore, A is connected.

Theorem 38
All intervals in $\mathbb{R}$ are connected.

Proof. We will prove this theorem by proving both directions.

($\Rightarrow$) Assume that I is an interval in $\mathbb{R}$. We will prove that I is connected.

($\Leftarrow$) Assume that I is connected. We will prove that I is an interval.

Let I be an interval in $\mathbb{R}$. We will prove that I is connected. Suppose, we have $U, V \subset I$, $U \cup V = I$, $U \cap V = \emptyset$. Since I is an interval, there exists a point $x \in I$ such that $x \in U$ and $x \in V$. Therefore, I is disconnected, which is a contradiction. Therefore, I is connected. $\square$

11 February 2, 2026

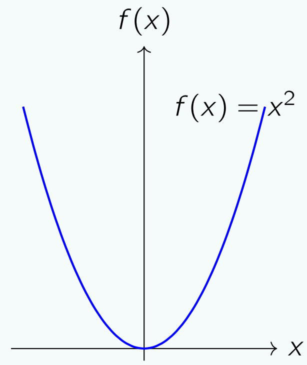
11 Continous Functions

Example 39

Consider the following functions:

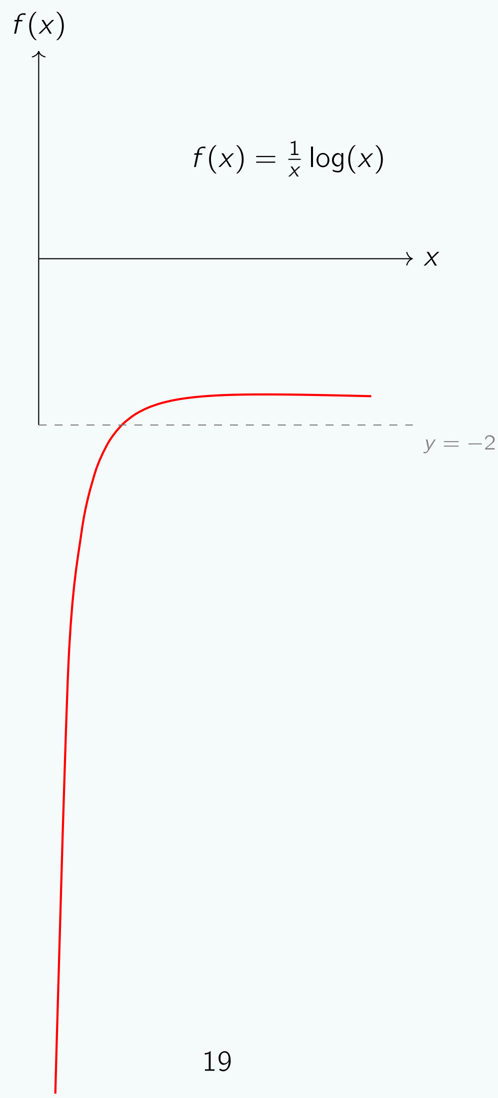
1. $f(x) = x^2$

   This function is continuous on $\mathbb{R}$. Its graph is the familiar parabola.



2. $f(x) = \frac{1}{x} \cdot \log(x)$

   This function is continuous on the interval $(0, \infty)$.

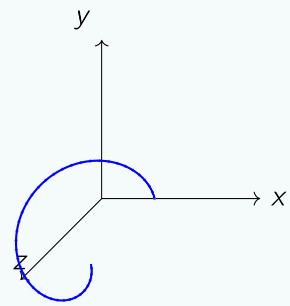


Example 40

Consider the space curve defined by the vector-valued function:

$$\mathbf{r}(t) = \langle \cos t,\ \sin t,\ t \rangle, \quad t \in \mathbb{R}$$

This is a parametrization of a **helix** in $\mathbb{R}^3$, which is a continuous map from $\mathbb{R}$ to $\mathbb{R}^3$.



(Visualization of $\mathbf{r}(t) = (\cos t, \sin t, t)$ for $0 \le t \le 2\pi$)

Definition 41 (Continuous Function)

Let E and E' be metric spaces, with distances denoted by d and d' respectively, let $f : E \to E'$ be a function. Then f is **continuous** at a point $p \in E$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $p_0 \in E$ with $d(p_0, p) < \delta$, we have $d'(f(p_0), f(p)) < \epsilon$. If f is continuous at every point in E, then f is **continuous on E**.

Example 42

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1$. We will prove that f is continuous at $p = 1$.

Proof. Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that for all $x \in \mathbb{R}$ with $d(x, 1) < \delta$, we have $d'(f(x), f(1)) < \epsilon$. Let $\delta = \epsilon$. Then for all $x \in \mathbb{R}$ with $d(x, 1) < \delta$, we have $d'(f(x), f(1)) = |f(x) - f(1)| = |1 - 1| = 0 < \epsilon$. Therefore, f is continuous at $p = 1$. $\square$

Example 43

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. We will prove that f is continuous at $p = 1$.

Proof. Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that for all $x \in \mathbb{R}$ with $d(x, 1) < \delta$, we have $d'(f(x), f(1)) < \epsilon$. Let $\delta = \epsilon$. Then for all $x \in \mathbb{R}$ with $d(x, 1) < \delta$, we have $d'(f(x), f(1)) = |f(x) - f(1)| = |x - 1| < \epsilon$. Therefore, f is continuous at $p = 1$. $\square$

Example 44

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. We will prove that f is continuous at $p = 1$.

Proof. Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that for all $x \in \mathbb{R}$ with $d(x, 1) < \delta$, we have $d'(f(x), f(1)) < \epsilon$. Let $\delta = \epsilon$. Then for all $x \in \mathbb{R}$ with $d(x, 1) < \delta$, we have $d'(f(x), f(1)) = |f(x) - f(1)| = |x^2 - 1| = |x - 1||x + 1| < \delta|x + 1| < \epsilon|x + 1| < \epsilon$. Therefore, f is continuous at $p = 1$. □

Example 45

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 2x - 3$. We will prove that f is continuous at $p = 1$.

Proof. Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that for all $x \in \mathbb{R}$ with $d(x, 1) < \delta$, we have $d'(f(x), f(1)) < \epsilon$. Let $\delta = \epsilon$. Then for all $x \in \mathbb{R}$ with $d(x, 1) < \delta$, we have $d'(f(x), f(1)) = |f(x) - f(1)| = |x^3 + 2x - 3 - (1^3 + 2 \cdot 1 - 3)| = |x^3 + 2x - 3 - 0| = |x^3 + 2x - 3| < \delta|x^2 + x + 1| < \epsilon$. Therefore, f is continuous at $p = 1$. □

Example 46

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$. We will prove that f is **not uniformly** continuous at $x_0 = 0$.

Proof. Suppose, we have $\epsilon > 0$ be given. Suppose, we have $\delta > 0$ such that for all $x \in \mathbb{R}$ with $d(x, 0) < \delta$, we have $d'(f(x), f(0)) < \epsilon$. Let $\delta = \epsilon$. Then for all $x \in \mathbb{R}$ with $d(x, 0) < \delta$, we have $d'(f(x), f(0)) = |f(x) - f(0)| = |0 - 1| = 1 > \epsilon$. Therefore, f is not uniformly continuous at $x_0 = 0$. □